AFSPACECOM TECHNICAL NOTE

SURFACE NAVIGATION AND GEODESY

A PARAMETRIC APPROACH

SPACE COMMAND

HEADQUARTERS AIR FORCE SPACE COMMAND

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A PARAMETRIC APPROACH

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HEADQUARTERS AIR FORCE SPACE COMMAND

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PREFACE

This document is the result of 2 years of independent research and study in the area of minimum distance theory and its application to geodesic measurement and surface navigation. The purpose is to develop a new approach to solving the problem of getting from point a to point b on well behaved surfaces. The study is confined to the use of curvilinear coordinates (u_1, u_2, u_3) and to surfaces which can be represented by a function, k, such that:

$u_3 = k(u_1, u_2)$, (ie, surfaces where one coordinate can be written as a function of the remaining two).

In rederiving the relations which determine the geodesics or curves of minimum distance, we shall see how using parametric separation of variables, along with the above coordinate variables and surface restrictions, greatly simplifies the problem, thus giving manageable solutions.

The paper addresses the development of solutions not just from the point of theoretical interest, but in terms of product useability, for eventual software implementation. By product useability, I mean we need to know more than just the relations between u_2 and u_3 for an arbitrary minima curve, we need explicit relations for distance traveled and direction (measured normally on the surface) of the minima curve. Thus, we consider a closed set of equations the solution for a given surface (and initial and/or boundary conditions) to be of the form:

$u_3 = k(u_1, u_2)$ (surface constraint).

$u_1 = f(\lambda)$ (parametric separation, first variable).

$u_2 = g(\lambda, a, b)$ (parametric minima solution, second variable).

$\Delta s = s(\lambda, a, b)$ (parametric distance traveled).

$\psi = \psi(\lambda, a, b)$ (azimuthal heading, 0-360, off a known constant/known direction).

where λ is the parametric separation variable, and a and b are constants of integrations which (assumably) uniquely determine the curve.

Several surfaces are addressed, but most rigorously treated is the sphere, where the second order, nonlinear differential equation which governs the extremum solution is solved for both boundary condition and initial condition cases, as both cases are of physical interest.

THIS TECHNICAL NOTE IS THE WORK OF DR. K. CARLTON-WIPPERN AND DOES NOT NECESSARILY REPRESENT THE OFFICIAL VIEW OF THE US AIR FORCE.

The nature of the oblateness of the earth is very briefly discussed, and a first order, perturbation approach (based on e^2, the square of the eccentricity) set of equations are developed. A comparison of the e^2 perturbed solutions to those of the unperturbed spherical solutions is discussed in appendix I. The e^2 perturbed solution set constitutes the mathematical foundation for Project Frost (First order formulae relating to oblate spheriod trigonometry).

While the purpose of this document is to provide useable products, not just address theoretical development, some attention has been paid to the existence and uniqueness of solutions, particularly in the spherical case, as there do exist traps for the unwary.

The application for the equations and solutions for the problems addressed in this document runs the gamut from commercial to governmental, particularly mapping agencies, and tactical and strategic defense organizations.

The mathematics developed herein are for use only as solutions to problems of surface geometry, curves of minimum distance. No attempt has been made to correct for physical phenomena, such as wind, geopotential, tides, currents etc. Corrections for heading and positioning are outside the scope of this study.

The format used is necessarily terse, and the document assumes the reader to be proficient in variational calculus, differential equations and linear algebra, as well as spherical trigonometry and in other applied mathematics/physics curricula.

CONTENTS

FIGURES

I. INTRODUCTION

The investigation begins with the general distance equation:

$$(ds,ds) = g_{ab}dx^a dx^b$$

or

$$s = \int_{t_0}^{t_1} \left(g\alpha\beta \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right)^{1/2} dt$$

for a space curve given by $x^i = x^i(t)$.

The curves of minimum distance (or geodesics) are found by solving:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \quad .$$

where

$$\Gamma^\tau_{\alpha\beta} = \frac{g^{\tau\sigma}}{2}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\sigma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \right)$$, called the Christoffel symbols of the second kind.[1]

For the purpose of this investigation, consider curvilinear coordinates only (which greatly simplifies things, reference Lass[1] or Morse and Feshbach[2]) also, the investigation primarily works with R^2, and assumes if u_1, u_2 and u_3 are curvilinear coordinates there exists a surface function, k, such that:

$$u_3 = k(u_1, u_2) \quad .$$

The distance formula becomes,

$$s = \int \left[h_1^2\left(\frac{du_1}{d\lambda}\right)^2 + h_2^2\left(\frac{du_2}{d\lambda}\right)^2 + h_3^2\left(\frac{du_3}{d\lambda}\right)^2 \right]^{1/2} d\lambda \qquad (1)$$

[1] Lass, Vector and Tensor Analysis, p. 288.

[1] Lass, op.cit., p. 281.

[2] Morse and Feshbach, Methods of Theoretical Physics, p. 47.

where h_1, h_2, and h_3 can be called scaling factors.[3]

Seek parametric solutions for u_1, u_2 which will minimize s, since

$$u_3 = k(u_1, u_2) \quad .$$

The solutions (for u_1, u_2) can be written parametrically of the form:

$$u_1 = f(\lambda) \quad ,$$

$$u_2 = g(\lambda) \quad .$$

f can be chosen arbitrarily (to facilitate integration), find $g(\lambda)$ via the usual methods of variational calculus,[4]

$$\text{Let } \delta^2 = \sum_{i=1}^{3} h_i^2 \left(\frac{du_i}{d\lambda}\right)^2$$

with $u_3 = k(f(\lambda), u_2)$, $u_1 = f(\lambda)$, then extremum is when:

$$\frac{d}{d\lambda}\left(\frac{\partial \delta}{\partial (u_2')}\right) - \frac{\partial \delta}{\partial u_2} = 0, \quad u_2' = \frac{du_2}{d\lambda}$$

letting $u_2 = g$,

$$\frac{h_3^2}{\delta} \cdot \frac{dk}{d\lambda} \cdot \frac{\partial}{\partial g}\left(\frac{dk}{d\lambda}\right) - \frac{d}{d\lambda}\left[\frac{1}{\delta}\left[h_2^2\left(\frac{dg}{d\lambda}\right) + h_3^2 \cdot \frac{dk}{d\lambda} \cdot \frac{\partial}{\partial g'}\left(\frac{dk}{d\lambda}\right)^2\right]\right] = 0 \quad . \qquad (2)$$

Rearranging 2a, taking a degree of freedom to facilitate integration, for example, assume there exists a function, f (= u_1), which sets

$$\frac{d}{d\lambda}\left[\frac{1}{\delta}\left(h_3^2 \cdot \frac{dk}{d\lambda} \cdot \frac{\partial}{\partial g}\left(\frac{dk}{d\lambda}\right)\right)\right] - \frac{1}{\delta} \cdot h_3^2 \cdot \frac{dk}{d\lambda} \cdot \frac{\partial}{\partial g}\left(\frac{dk}{d\lambda}\right) = 0 \qquad (2a)$$

[3] Spiegel, Advanced Mathematics for Engineers and Scientists, p. 288.

[4] Courant-Hilbert, Methods of Mathematical Physics, p. 164.

then taking this degree of freedom for f leaves

$$\frac{d}{d\lambda}\left(\frac{h_2^2 g'}{\delta}\right) = 0 \qquad (2b)$$

$$h_2^2 g' = c\delta$$

$$h_2^4 g'^2 = c^2\left[h_1^2\left(\frac{df}{d\lambda}\right)^2 + h_2^2\left(g'^2\right) + h_3^2\left(\frac{dk}{d\lambda}\right)^2\right] \qquad (3)$$

but

$$\frac{dk}{d\lambda} = \frac{\partial k}{\partial \lambda} + \frac{\partial k}{\partial g} g'$$

$$h_2^4 g'^2 - c^2 h_2^2 g'^2 = c^2\left\{h_1^2\left(\frac{df}{d\lambda}\right)^2 + h_3^2\left[\left(\frac{\partial k}{\partial \lambda}\right)^2 + \left(\frac{\partial k}{\partial g}\right)^2 g'^2 + 2\,\frac{\partial k}{\partial \lambda}\cdot\frac{\partial k}{\partial g}g'\right]\right\}. \qquad (3a)$$

Rearranging and collecting terms:

$$Ag'^2 + 2Bg' + C = 0 \qquad (4)$$

$$A(\lambda,g) = c^2\left(h_2^2 + h_3^2\left(\frac{\partial k}{\partial g}\right)^2\right) - h_2^4$$

$$B(\lambda,g) = c^2 h_3^2\left(\frac{\partial k}{\partial \lambda}\right)\left(\frac{\partial k}{\partial g}\right)$$

$$C(\lambda,g) = c^2\left[h_1^2\left(\frac{\partial f}{\partial \lambda}\right)^2 + h_3\left(\frac{\partial k}{\partial \lambda}\right)^2\right].$$

Now calling

$$\eta(\lambda,g) = -\frac{1}{A}\left\{B \pm \left[B^2 - AC\right]^{1/2}\right\}.$$

And assuming η can be uniquely determined by the boundary, surface/other logical elimination conditions, then

$$g' = \eta(\lambda,g) \quad . \qquad (5)$$

The geodesic curves are found by choosing f to make equation 3 possible, then solving the integral equation:

$$g(\lambda) = \int \eta(\lambda,g)\, d\lambda \quad . \qquad (6)$$

This can be considered a navigation kernal, in that for a surface defined by k, the constraint condition for f, equation (2a) and the solution to equation (6) gives us the parametric equations of the path(s) of minimum distance, subject to the boundary conditions (end points of the curve on the surface).

While traveling on a curve of minimum distance, it is convenient to know in what direction to go to remain on the curve. Using the natural definition of the tangent vector $\left(= \frac{d\vec{R}}{ds}\right)$ aids getting around on the curve. If,

$u_1 = f(\lambda)$

$u_2 = g(\lambda)$

$u_3 = k(\lambda)$, and if $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are curvilinear unit vectors, then:

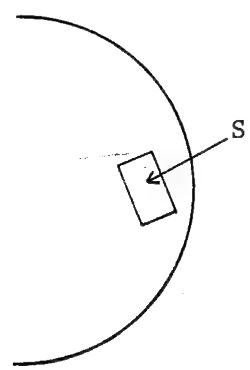
$$\vec{D} = h_1 \frac{\partial f}{\partial \lambda} \hat{e}_1 + h_2 \frac{\partial g}{\partial \lambda} \hat{e}_2 + h_3 \frac{\partial k}{\partial \lambda} \hat{e}_3 \quad .$$

(Compare this result to the contravariant vector.[5])

If the surface on which the navigation is taking place can be written as $S(u_1, u_2, u_3) = 0$, and the parametric curve of minimum distance can be written also as a surface $\zeta(u_1, u_2, u_3) = 0$ such that the two surfaces intersect in the space curve of minimum distance, then the heading vector $\vec{H}$ can be written as:

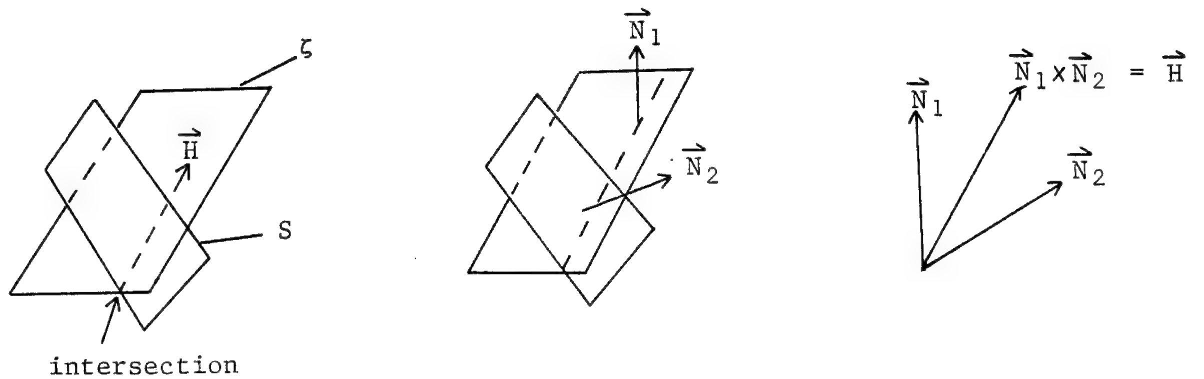
$$\vec{H} = \vec{\nabla}\zeta \times \vec{\nabla}S \quad .$$

To demonstrate this, let S, the surface on which the navigation is taking place, be represented as locally (approximately) planar.



Similarly, let ζ, the surface associated with the minimum distance constraint, be approximately planar (this is roughly equivalent to using affine approximations to the actual surfaces about a small, differential area of local intersection, assuming the intersection exists).

[5] Lass, op.cit., pgs. 270-272.



The intersection (of the two planes) is a line. Let $\vec{N}_2$ be the surface normal to S and $\vec{N}_1$ be the surface normal to ζ. the direction of $\vec{H}$ (if not $\equiv 0$) is clearly $\vec{H} = \vec{N}_1 \times \vec{N}_2$ by construction. In the limit that the differential planar surface approximations go to zero $\vec{N}_2 \to \vec{\nabla}S$, $\vec{N}_1 \to \vec{\nabla}\zeta$, so that $\vec{H} = \vec{\nabla}\zeta \times \vec{\nabla}S$ (gradients to be evaluated at a point of interest on S).

So long as $\vec{H} \equiv 0$, the magnitude of $|H|$ is relatively unimportant. It is the direction of $\vec{H}$ which matters. Eventually, $|\vec{H}|$ drops out of the analysis entirely, it is from the components of $\vec{H}$ that heading angles are determined. (The same argument is true for $\vec{D}(\lambda)$.)

II EXAMPLE IN CARTESIAN COORDINATES

a. Application to the planar surface.

$$s = \int \left(\left(\frac{dx}{d\lambda} \right)^2 + \left(\frac{dy}{d\lambda} \right)^2 + \left(\frac{dz}{d\lambda} \right)^2 \right)^{1/2} d\lambda \quad .$$

For this example the surface is defined as a plane.

$z = a + bx + cy$

and search for $g(\lambda)$ such that the integral is extrema. For example, let,

$x = \lambda$

$y = g(\lambda)$

$z = a + b\lambda + cg(\lambda)$,

then

$$s = \int \left[1 + \left(g' \right)^2 + \left(b + cg' \right)^2 \right]^{1/2} d\lambda$$

$$g' = \frac{dg}{d\lambda} \; .$$

With δ = integrand, note that $\frac{\partial \delta}{\partial g} = 0$; therefore,

$\frac{d}{d\lambda}\left(\frac{\partial \delta}{\partial g'}\right) = 0$, and after integrating once:

$$e = \frac{g' + c\left(b + cg'\right)}{\left[1 + g'^2 + \left(b + cg'\right)^2\right]^{1/2}} \qquad \text{(e is a constant of integration).}$$

Continuing with:

$$\left[1 + g'^2 + \left(b + cg'\right)^2\right] e^2 = \left[g' + c\left(b + cg'\right)\right]^2 \; .$$

This is a quadratic equation in g', of the form:

$Ag'^2 + Bg' + C = 0$ (A, B and C are constants).

Now, solving the equation for g', let

$g' = D$, assuming D to be uniquely determinable from the geometry,

then

$g(\lambda) = D\lambda + E$

and substituting back

$x = \lambda$

$y = D\lambda + E$

$z = F\lambda + G \qquad [= a + b\lambda + c(D\lambda + E)]$.

This is the parametric equation for a line in R^3.

If the line passes through points $\left(x_0, y_0, z_0\right)$ and $\left(x_1, y_1, z_1\right)$

then,

$x = \left(x_1 - x_0\right)\lambda + x_0$

$y = \left(y_1 - y_0\right)\lambda + y_0$

$z = \left(z_1 - z_0\right)\lambda + z_0$

is an appropriate expression.

Letting $a = x_1 - x_0$, $b = y_1 - y_0$, $c = z_1 - z_0$

then,

$\vec{D} = a\hat{i} + b\hat{j} + c\hat{k}$*

and for distance

$$\Delta s(\lambda) = \left(a^2 + b^2 + c^2\right)^{1/2} \Delta\lambda \quad .$$

The parametric equation can also be written

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \lambda + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

or solving individually for λ, then setting components equal to each other

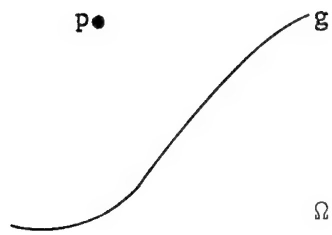
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad .$$

The forms are equivalent.

*$\vec{D}$, a tangential vector $\left(\text{the tangential vector is often referred to as } \vec{T} = \frac{d\vec{r}}{ds}\text{, so that } \vec{T} = \frac{\vec{D}}{|\vec{D}|}\right)$, this is equivalent to saying ψ = constant on the reference surface, where ψ is a heading angle measured on the surface, referenced from some known directional vector.

b. <u>Minimum distance solution.</u>

Consider the following diagram



To find the minimum distance from p to the curve g, let Ω be a plane (in particular, the x, y plane). If $\hat{f}$ is the solution to the minimum distance problem on a plane, $\hat{f}$ is a line of the form $y = ax + b$ or $x = a'y + b'$, then the minimum distance between p and g lies along a curve $\hat{f}$, hence if $p = \left(x_0, y_0 \right)$, letting x be the independent variable, then,

$$\left(\Delta s \right)^2 = \left(x - x_0 \right)^2 + \left(g(x) - y_0 \right)^2$$

and this is extremum when $\frac{ds}{dx} = 0$.

$$0 = \left(x - x_0 \right) + \left(g(x) - y_0 \right) \frac{dg}{dx} \qquad (7)$$

so that

$$\frac{dg}{dx} = - \frac{\left(x - x_0 \right)}{\left(g(x) - y_0 \right)} \; . \qquad (8)$$

Solving for x (on g) in equation (7), let the result be x_1, then find $y_1 = g\left(x_1 \right)$ $\left[\text{assuming } g\left(x_1 \right) \neq y_0 \right]$ and, constructing a line from $\left(x_0, y_0 \right)$ to $\left(x_1, y_1 \right)$ solve for Δs. Equation (8) indicates that the line from $\left(x_0, y_0 \right)$ to $\left(x_1, y_1 \right)$ intersects g at right angles. There may be more than one solution to equation (7), in which case the absolute (instead of relative) minimum would have to be found, assuming one existed. Whatever the case, the solution, by equation (8), guarantees that the line segment intersects g as a perpendicular.

In that a smooth, well behaved surface can be represented as a limiting number of planes, (ie, a surface S can be approximated by n, $\lim_{n\to\infty}$ differential planar surfaces or affine surface approximations) it seems plausable that if g = $\hat{f}$, $\hat{f}$ a minima curve on a surface S, then if p is a point on the surface but not on $\hat{f}$, and if $\hat{f}'$ is another minima curve which goes through p and intersects $\hat{f}$ at right angles, then $\hat{f}'$ is the curve of minimum distance between p and $\hat{f}$. As will be shown in the spherical development, $\hat{f}'$ need not necessarily be unique for the general case. Stated mathematically

Let $p \in \hat{f}$

and let $\hat{f} = \hat{f}(\lambda)$

and

$\hat{f}' = \hat{f}'(\lambda)$*

Then $\hat{f}$ minimizes $\int ds$ between p and $\hat{f}$ if there exists λ_0 such that

$\hat{f}(\lambda_0) = \hat{f}'(\lambda_0)$, = p

and

$$\vec{D}(\hat{f}) \cdot \vec{D}(\hat{f}') = 0 \qquad \left(\hat{f},\ \hat{f}' \text{ evaluated at } \lambda_0\right) .$$

Now with $\vec{D}(\hat{f}) = h_1 \frac{da}{d\lambda}\hat{e}_1 + h_2 \frac{db}{d\lambda}\hat{e}_2 + h_3 \frac{dc}{d\lambda}\hat{e}_3$**

hence:

$$h_1^2 \frac{da}{d\lambda} \cdot \frac{da'}{d\lambda} + h_2^2 \frac{db}{d\lambda} \cdot \frac{db'}{d\lambda} + h_3^2 \frac{dc}{d\lambda} \cdot \frac{dc'}{d\lambda} \Big| = 0$$

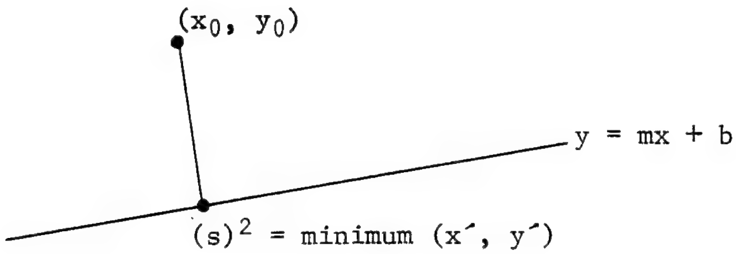
(apriori uniqueness not guaranteed.)

*ie, $\hat{f} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} a(\lambda) \\ b(\lambda) \\ c(\lambda) \end{pmatrix}$, $\hat{f}' = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} a'(\lambda) \\ b'(\lambda) \\ c'(\lambda) \end{pmatrix}$.

It must be remembered that this case is concerned with solutions to second order differential equations with boundary conditions. Depending on the surface geometry and the boundary conditions, the minima solutions may not be unique, as is shown in the spherical section.

The Cartesian example is finished by working out explicity the point to line on a plane example.

**Compare this with the definition of the contravariant vector, note also $ds = (\vec{D} \cdot \vec{D})^{1/2} d\lambda$.



$$(\Delta s)^2 = (x_0 - x)^2 + (y_0 - [mx + b])^2$$

$$\frac{ds}{dx} = 0 = -(x_0 - x) + (-m)(y_0 - [mx + b])$$

hence,

$$x = \frac{x_0 + my_0 - mb}{1 + m^2}, \text{ call this point } x' \quad .$$

$$y' = m(x') + b = \frac{mx_0 + m^2 y_0 + b}{1 + m^2} \quad .$$

$$\text{Slope of intersecting line} = \frac{y' - y_0}{x' - x_0}$$

$$= \frac{\dfrac{mx_0 + m^2 y_0 + b}{1 + m^2} - y_0}{\dfrac{x_0 + my_0 - mb}{1 + m^2} - x_0} = \frac{mx_0 - my_0 + b}{-m^2 x_0 + my_0 - mb} = -\frac{1}{m} \quad .$$

NOTE: Slope of $y = m$, slope of intersecting line $= -\frac{1}{m}$. QED.

III EXAMPLES IN CYLINDRICAL COORDINATES

a. Application to the cylindrical surface.

$$s = \int \left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\theta}{d\lambda}\right)^2 \right]^{1/2} d\lambda \quad .$$

For the first example define the surface constraint to be that of a cylinder.

$r = a$.

Search for $\theta = g(\lambda)$ such that the integral is minima.

Letting $z = a\lambda$, obtain

$$s = a\int \delta \, d\lambda$$

where,

$$\delta = \left[1 + \left(\frac{dg}{d\lambda}\right)^2\right]^{1/2} \quad .$$

Noting $\frac{\partial \delta}{\partial g} = 0$,

then,

$\frac{d}{d\lambda}\left(\frac{\partial \delta}{\partial g'}\right) = 0$, and integrating once

$$g'\left(1 + g'^2\right)^{-1/2} = c$$

$$g'^2 = c^2 + c^2 g'^2 \quad , g' = \beta \quad , \beta = \pm\frac{c}{\left(1 - c^2\right)^{1/2}}$$

and

$$g(\lambda) = \beta\lambda + \theta_0 \quad , \theta(\lambda) = \theta_0 + \beta\lambda \quad .$$

The solution in parametric format

$$r = a$$

$$z = a\lambda$$

$$\theta = \theta_0 + \beta\lambda$$

$\left[\text{also } z = \frac{a}{\beta}\left(\theta - \theta_0\right) \text{ after eliminating } \lambda\right]$.

Now,

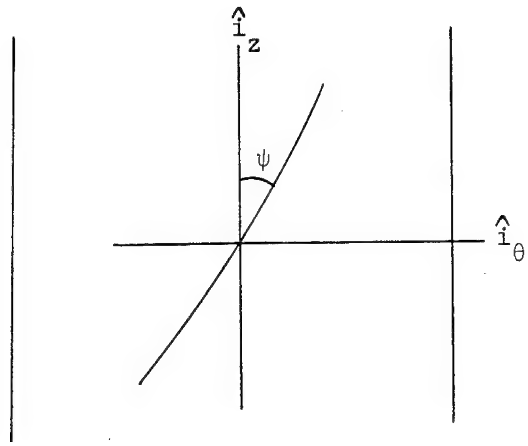
$s = \int \left(1 + \beta^2\right)^{1/2} d\lambda$ solves as

$$\Delta s = a\left(1 + \beta^2\right)^{1/2} \cdot \Delta\lambda \Big|_{\lambda_0}^{\lambda_1} \quad .$$

Letting $\hat{e}_1 = \hat{i}_z$, $\hat{e}_2 = \hat{i}_r$, $\hat{e}_3 = \hat{i}_\theta$

$\vec{D} = a\hat{i}_z + r\beta\hat{i}_\theta = a\hat{i}_z + a\beta\hat{i}_\theta$.

NOTE: $\vec{D}$ = Constant, independent of λ.



ψ is defined as $\tan\psi = \dfrac{\vec{D}\cdot\hat{i}_\theta}{\vec{D}\cdot\hat{i}_z} = \beta$, ie, ψ is a constant.

Similarly, the surface function is defined as S: $(r - a) = 0$ and the minimum distance function is f: $-z + \frac{a}{\beta}\left(\theta - \theta_0\right) = 0$,

then,

$\vec{H} = \vec{\nabla}f \cdot \vec{\nabla}S$

$\vec{\nabla}f = -1\hat{i}_z + \frac{1}{r}\left(\frac{a}{\beta}\right)\hat{i}_\theta = -\hat{i}_z + \frac{1}{\beta}\hat{i}_\theta$

$\vec{\nabla}S = \hat{i}_r$ hence,

$$\vec{H}\begin{vmatrix} \hat{i}_z & \hat{i}_r & \hat{i}_\theta \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{\beta} \end{vmatrix} = \frac{1}{\beta}\hat{i}_z + \hat{i}_\theta \quad .$$

This results in $\tan\psi = \dfrac{\vec{H}\cdot\hat{i}_\theta}{\vec{H}\cdot\hat{i}_z} = \beta$.

NOTE: The heading angle ψ is easily obtained when either $\vec{D}$ or $\vec{H}$ can be written in only two coordinate directions, that is to say, given curvilinear u_1, u_2, u_3, if the surface can be written such that u_k = constant, ψ is easily found by dot product of $\vec{D}$ or $\vec{H}$ by the remaining unit vectors $\hat{e}_{\hat{i}}$, $\hat{e}_{\hat{j}}$ $(i,j \neq k)$. If not, rotate $\vec{D}$ or $\vec{H}$ as necessary to produce a meaningful result (reference IVb for additional data on the heading angle).

The five equation solution for the cylinder is:

$$r(\lambda) = a$$

$$z(\lambda) = a\lambda$$

$$\theta(\lambda) = \theta_0 \pm \beta\lambda$$

$$\Delta s(\lambda) = a\left(1 + \beta^2\right)^{1/2}\lambda \Big|_{\lambda_0}^{\lambda_1}$$

$$\psi(\lambda) = \tan^{-1}\beta \quad .$$

b. Application to the spherical surface. Another example in cylindrical coordinates is as follows:

$$s = \int\left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda$$

The surface constraint is defined to be that of a sphere,

$$r^2 + z^2 = a^2, \; \left[z = \pm\left(a^2 - r^2\right)^{1/2}\right] \quad .$$

Choosing $r^2 = \dfrac{a^2}{1 + \lambda^2}$, which is obtained from the following analysis:

From,

$$s = \int\left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda$$

and since $z = f(r)$ only

$$s = \int\left[\left(1 + \left(\frac{dz}{dr}\right)^2\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda \quad .$$

Then with $r^2 + z^2 = a^2$

$$\frac{dz}{dr} = -\frac{r}{z}$$

$$1 + \left(\frac{dz}{dr}\right)^2 = \frac{a^2}{a^2 - r^2} \quad .$$

Letting $r = af$, $r^2 = a^2 f^2$, $\theta = g$, $\frac{d\theta}{d\lambda} = g'$

$$s = a\int\left\{\frac{1}{\left(1 - f^2\right)} f'^2 + f^2 g'^2\right\}^{1/2} d\lambda \quad .$$

Define the integrand as δ, and by inspection, $\frac{\partial\delta}{\partial g} = 0$, so, already integrating

once, $\frac{\partial\delta}{\partial g'} = c$

$$f^2 g' = c\left[\frac{f'^2}{1 - f^2} + f^2 g'^2\right]^{1/2}$$

$$g'^2 = c^2\left[\frac{f'^2 f^{-4}}{1 - f^2} + f^{-2} g'^2\right] \quad .$$

Take a degree of freedom by choosing f such that,

$$\frac{f'^2 f^{-4}}{1 - f^2} = 1$$

$$f' = f^2\left(1 - f^2\right)^{1/2}$$

$$\frac{df}{f^2\left(1 - f^2\right)^{1/2}} = d\lambda$$, let $f = \sin u$, $df = \cos u du$

$$\csc^2 u du = d\lambda$$

$$-\cot u = \lambda$$

$$-\frac{\left(1 - f^2\right)^{1/2}}{f} = \lambda$$

$$\frac{1 - f^2}{f^2} = \lambda^2 \quad , \frac{1}{f^2} = \lambda^2 + 1 \quad , f^2 = \frac{1}{1 + \lambda^2} \quad .$$

Now, since $r = af$, $r^2 = a^2f^2$, $r^2 = \dfrac{a^2}{1 + \lambda^2}$, then continuing:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{a^2 r^2}{\left(1 + \lambda^2\right)^3}$$

$$\left(\frac{dz}{d\lambda}\right)^2 = \left(\frac{dz}{dr}\ \frac{dr}{d\lambda}\right)^2 = \frac{a^2}{\left(1 + \lambda^2\right)^3}$$

$$s = a\int \delta\, d\lambda$$

$$\delta^2 = \frac{\lambda^2}{\left(1 + \lambda^2\right)^3} + \frac{1}{\left(1 + \lambda^2\right)^3} + \frac{\left(g'\right)^2}{1 + \lambda^2} = \frac{1}{\left(1 + \lambda^2\right)^2} + \frac{\left(g'\right)^2}{1 + \lambda^2}$$

$$\frac{\partial\delta}{\partial g} = 0 \quad , \frac{\partial\delta}{\partial g'} = c \quad \text{(integrating once).}$$

$$g'^2 = c^2 + c^2 g'^2 \left(1 + \lambda^2\right), \quad \beta^2 = \frac{c^2}{1 - c^2}$$

$$g' = \pm \frac{\beta}{\left(1 - \beta^2\lambda^2\right)^{1/2}}$$

$$g = \theta_0 \pm \cos^{-1} \beta\lambda \quad .$$

Now, parametrically:

$$r = \frac{a}{\left(1 + \lambda^2\right)^{1/2}} \quad \text{(assuming } r \geqslant 0\text{)}$$

$$\theta = \theta_0 \pm \cos^{-1} \beta\lambda$$

$$z = \pm \frac{a\lambda}{\left(1 + \lambda^2\right)^{1/2}} \quad .$$

To eliminate λ, start with

$$\lambda = \beta^{-1} \cos\left(\theta - \theta_0\right) = \beta^{-1} \cos\left(\theta_0 - \theta\right)$$

then choosing,

$$r^2 = \frac{a^2}{1 + \beta^{-2} \cos^2\left(\theta - \theta_0\right)}$$

continue with,

$$r^2 + r^2\beta^{-2} \cos^2\left(\theta - \theta_0\right) = a^2$$

rearranging and multiplying by ξ^2

$$r^2\left(\frac{\xi}{\beta'}\right)^2 \cos^2\left(\theta - \theta_0\right) = \xi^2 z^2 \qquad (z^2 = a^2 - r^2)$$

hence,

$$r\left(\frac{\xi}{\beta}\right)\cos\left(\theta - \theta_0\right) = \xi z \quad .$$

Collecting factors and rearranging to Cartesian coordinates, obtain $ax + by + cz = 0$ which is the equation of a plane confined to contain the origin.

NOTE: $x = y = z = 0$ <u>always</u> a solution.

c. <u>Application to the conical surface</u>. Another example in cylindrical coordinates.

$$s = \int \left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} \quad .$$

Define the surface constraint to be that of a right cone:

$$r = \alpha z \quad ,$$

let,

$$r = a\lambda$$

$$s = a\int\left[\gamma^2 + \lambda^2 g'^2\right]^{1/2} d\lambda \qquad \left(\gamma^2 = 1 + \frac{1}{\alpha^2}\right)$$

as before

$$\frac{\partial\delta}{\partial g} = 0 \quad , \quad \frac{\partial\delta}{\partial g'} = c \quad ,$$

then rearranging:

$$\lambda^4 g'^2 = c^2\gamma^2 + c^2\lambda^2 g'^2$$

$$g = \gamma\int\frac{df}{f\left(f^2 - 1\right)^{1/2}} \qquad (\lambda = fc \quad , \; d\lambda = cdf)$$

$$g = \theta_0 \pm \gamma \sec^{-1} f$$

hence:

$$r = a\lambda$$

$$z = \frac{a}{\alpha}\lambda$$

$$\theta = \theta_0 + \gamma \sec^{-1}\frac{\lambda}{c}$$

$$\Delta s = a\int\left[\gamma^2 + \frac{c^2\gamma^2}{\lambda^2 - c^2}\right]^{1/2} d\lambda$$

$$= a\gamma\int\frac{\lambda\, d\lambda}{\left(\lambda^2 - c^2\right)^{1/2}}$$

$$= a\gamma\left(\lambda^2 - c^2\right)^{1/2}\Big|_{\lambda_0}^{\lambda_1} \qquad (\lambda \geqslant c).$$

IV EXAMPLES IN SPHERICAL COORDINATES

a. Application to the conical surface.

$$\left(ds\right)^2 = \left(d\rho\right)^2 + \rho^2\left(d\phi\right)^2 + \rho^2\sin^2\phi\left(d\theta\right)^2.$$

The surface constraint is defined to be that of a right cone, ϕ = constant, which can be defined as

$$\phi = \tan^{-1}\alpha \quad .$$

Letting

$$\rho = a\lambda$$

and

$$\sin^2\phi = \gamma^2 = \frac{\alpha^2}{1 + \alpha^2}$$

continue with:

$$s = a\int\left(1 + \gamma^2\lambda^2 g^{-2}\right)^{1/2} d\lambda$$

$$\frac{\partial \delta}{\partial g} = 0 \quad ,$$

$$\frac{\partial \delta}{\partial g'} = c$$

and

$$c^2 = \gamma^2 \lambda^2 \left(\gamma^2 \lambda^2 - c^2 \right) g'^2$$

$$g' = \pm \frac{c}{\gamma\lambda} \left(\gamma^2 \lambda^2 - c^2 \right)^{-1/2}$$

$$g(\lambda) = \theta_0 \pm \frac{1}{\gamma} \sec^{-1} \frac{\gamma\lambda}{c} \quad .$$

(1) Conic distance measurement.

$$\Delta s = a \int \left(1 + \frac{c^2}{d^2 \lambda^2 - c^2} \right)^{-1/2} d\lambda = a \int \frac{\gamma \lambda d\lambda}{\left(\gamma^2 \lambda^2 - c^2 \right)^{1/2}}$$

$$\Delta s = a \left(\lambda^2 - \frac{c^2}{\gamma^2} \right)^{1/2} = \frac{a}{\gamma} \left(\lambda^2 \gamma^2 - c^2 \right)^{1/2} \quad .$$

(2) Conic heading vector development.

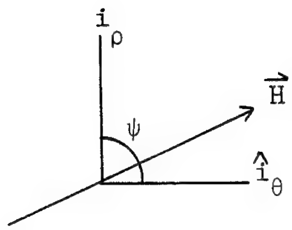
$$s = \phi - \tan^{-1} \alpha$$

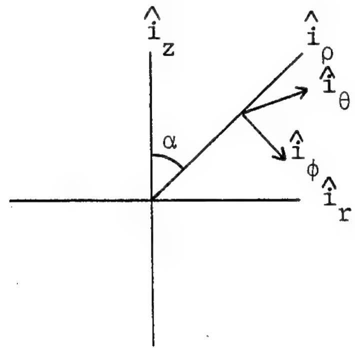
$$f = -\frac{ac}{\gamma} \sec \gamma \left(\theta - \theta_0 \right) + \rho \qquad \text{(choosing the positive direction).}$$

$$\vec{H} = \vec{\nabla} f \times \vec{\nabla} s = \begin{pmatrix} i_\rho & i_\phi & i_\theta \\ 1 & 0 & -\dfrac{ac}{\rho \sin \phi} \sec u \tan u \\ 0 & \dfrac{1}{\rho} & 0 \end{pmatrix} \qquad \left(u = \gamma \left(\theta - \theta_0 \right) \right)$$

$$\vec{H} = \left(\frac{ac}{\rho^2 \sin \phi} \sec u \tan u \right) \hat{i}_\rho + \frac{1}{\rho} \hat{i}_\theta \quad .$$

Now, looking on the conic surface:





$$\cos\psi = |\vec{H}|\,\frac{ac\sec u\tan u}{\rho^2\sin\phi} \qquad \left(\sin\psi = |\vec{H}|\,\frac{1}{\rho}\right)$$

$$\tan\psi = \frac{\rho\sin\phi}{ac\sec u\tan u} \quad .$$

Now,

$\sin\phi = \gamma,$

and from f:

$$\frac{ac\sec\gamma\left(\theta - \theta_0\right)}{\rho} = \gamma \quad .$$

Thus,

$$\tan\psi = \frac{1}{\tan u} = \cot\gamma\left(\theta - \theta_0\right) \quad .$$

Now, to show the equivalence between $\vec{H}$ and $\vec{D}$, choose

$\rho = a\lambda$

$\phi = \tan^{-1}\alpha$

$\theta = \theta_0 + \frac{1}{\gamma}\sec^{-1}\frac{\gamma\lambda}{c}$

obtain,

$$\vec{D} = a\hat{i}_\rho + \rho\emptyset\hat{i}_\phi + \rho\sin\phi\left(\frac{d\theta}{d\lambda}\right)\hat{i}_\theta \qquad (\rho = a\lambda \quad , \sin\phi = \gamma)$$

$$\vec{D} = a\left[\hat{i}_\rho + \frac{\lambda \frac{\gamma}{c}}{\frac{\gamma\lambda}{c}\left(\frac{\gamma^2\lambda^2}{c^2} - 1\right)^{1/2}}\ \hat{i}_\theta\right]$$

$$= a\left[\hat{i}_\rho + \frac{1}{\left(\frac{\gamma^2\lambda^2}{c^2} - 1\right)^{1/2}}\ \hat{i}_\theta\right] .$$

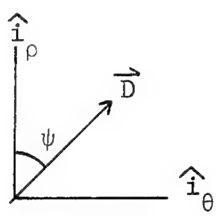
Now,

$$\vec{D} \cdot \hat{i}_\rho = a = \cos\psi\ |D|$$

$$\vec{D} \cdot \hat{i}_\theta = a\left(\frac{\gamma^2\lambda^2}{c^2} - 1\right)^{1/2} = \sin\psi\ |D|$$

hence,

$$\tan\psi = \left[\left(\frac{\gamma\lambda}{c}\right)^2 - 1\right]^{-1/2} = \left(\sec^2\left[\gamma\left(\theta - \theta_0\right)\right] - 1\right)^{-1/2} = \cot\gamma\left(\theta - \theta_0\right) \text{ QED.}$$



Notice when the surface constraint can be written u_k = constant, the coordinate system can be considered surface normal. In such a system, when $\vec{D}$ is evaluated, only two components are present. These components are along orthogonal topocentric unit vectors on the reference surface. The five equation solution for the cone is:

$$\rho(\lambda) = a\lambda$$

$$\phi(\lambda) = \tan^{-1}\alpha$$

$$\theta(\lambda) = \theta_0 \pm \frac{1}{\gamma}\sec^{-1}\left(\frac{\gamma\lambda}{c}\right) \qquad \left(\text{where } \gamma^2 = \frac{\alpha^2}{1+\alpha^2}\right).$$

$$\Delta s(\lambda) = a\left(\lambda^2 - \frac{c^2}{\gamma^2}\right)^{1/2}\Bigg|_{\lambda_0}^{\lambda_1} \qquad \left(\lambda \geq \frac{c}{\gamma}\right)$$

$$\psi(\lambda) = \tan^{-1}\left[\left(\frac{\gamma\lambda}{c}\right)^2 - 1\right]^{-1/2} .$$

b. Application to the spherical surface. Define the surface to be that of a sphere:

$\rho = a'$ (constant) and search for $\theta = g(\lambda)$ such that the integral is a minima.

$$s = \int\left[\left(\frac{d\rho}{d\lambda}\right)^2 + \rho^2\left(\frac{d\phi}{d\lambda}\right)^2 + \rho^2 \sin^2\phi\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda$$

(1) Solution by parametric separation of variables. The differential equation for spherical integrating factor is obtained from,

$$\Delta s = a'\int\left(\left(\frac{d\phi}{d\lambda}\right)^2 + \sin^2\phi\left(\frac{d\theta}{d\lambda}\right)^2\right)^{1/2} d\lambda .$$

Let,

$\phi = f$

$\theta = g$

and obtain:

$$\Delta s = a\int\left(f'^2 + \sin^2 f g'^2\right)^{1/2} d\lambda .$$

Noting,

$$\frac{\partial\delta}{\partial g} = 0 ,$$

then, integrating once:

$$\frac{\partial\delta}{\partial g'} = c = \frac{\sin^2 f g'}{\left(f'^2 + \sin^2 f g'^2\right)^{1/2}}$$

$$c^2 f^2 + c^2 \sin^2 f g'^2 = \sin^4 f g'^2$$

$$c^2 \frac{f'^2}{\sin^4 f} + c^2 \frac{g'^2}{\sin^2 f} = g'^2$$

$$c^2\left(\frac{f'}{\sin^2 f}\right)^2 + c^2 \frac{g'^2}{\sin^2 f} = g'^2 .$$

Choose f such that:

$$\frac{f'}{\sin^2 f} = \frac{1}{\lambda^2}$$

$$\frac{df}{\sin^2 f} = \frac{d\lambda}{\lambda^2}$$

$$\csc^2 f \, df = \frac{1}{\lambda}$$

$$-\cot f = -\frac{1}{\lambda}$$

$$f = \tan^{-1} \lambda$$

so that

$$\phi = \tan^{-1} \lambda \qquad (\lambda = \tan \phi).$$

Then,

$$\frac{d\rho}{d\lambda} = 0 \quad ,$$

$$\frac{d\phi}{d\lambda} = \frac{1}{1 + \lambda^2} \quad ,$$

$$\sin^2 \phi = \frac{\lambda^2}{1 + \lambda^2} \quad ,$$

$$\theta = g(\lambda)$$

$$s = a' \int \delta \, d\lambda$$

where,

$$\delta = \left[\left(1 + \lambda^2\right)^{-2} + \lambda^2 \left(1 + \lambda^2\right)^{-1} \left(\frac{dg}{d\lambda}\right)^2 \right]^{1/2} \quad .$$

Noting,

$$\frac{\partial \delta}{\partial g} = 0 \quad ,$$

$$\frac{d}{d\lambda}\left(\frac{\partial \delta}{\partial g'}\right) = 0$$

and integrating once gives

$$\frac{\lambda^2}{1+\lambda^2} g' = c'\delta \qquad c' = \text{constant of integration}$$

$$\left(\frac{\lambda^2}{1+\lambda^2}\right)^2 g'^2 = c'^2\left[\left(\frac{1}{1+\lambda^2}\right)^2 + \frac{\lambda^2}{\left(1+\lambda^2\right)} g'^2\right]$$

$$\lambda^4 g'^2 = c'^2 + c'^2\lambda^2\left(1+\lambda^2\right) g'^2$$

$$\lambda^4 g'^2 - c'^2\lambda^2 g'^2 - c'^2\lambda^4 g'^2 = c'^2$$

$$\left(1 - c'^2\right)\lambda^4 g'^2 - c'^2\lambda^2 g'^2 = c'^2$$

$$\lambda^2 g'^2\left[\left(1 - c'^2\right)\lambda^2 - c'^2\right] = c'^2$$

$$g' = \frac{\pm c'}{\lambda\left[\left(1 - c'^2\right)\lambda^2 - c'^2\right]^{1/2}} \quad .$$

Setting

$$\beta^{-2} = \frac{1 - c'^2}{c'^2}$$

and

$$q = \frac{\lambda}{\beta}$$

$$dq = \frac{1}{\beta}\, d\lambda$$

$$g(q) = \pm \int \frac{dq}{q\left(q^2 - 1\right)^{1/2}}$$

$$g(q) = \theta_0 \pm \sec^{-1} q$$

$$g(\lambda) = \theta_0 \pm \sec^{-1}\frac{\lambda}{\beta} \qquad \left(\text{with } \theta_0 \text{ and } \beta \text{ constants of integration}\right).$$

Parametrically, the curve is

$$\rho = a'$$

$$\phi = \tan^{-1}\lambda$$

$$\theta = \theta_0 \pm \sec^{-1} \frac{\lambda}{\beta} \quad .$$

To gain further insight as to the nature of this curve, work backwards as follows:

$$\theta - \theta_0 = \sec^{-1} \frac{\lambda}{\beta}$$

$$\lambda = \beta \sec\left(\theta - \theta_0\right)$$

$$\lambda = \frac{-c}{-c\beta^{-1} \cos\left(\theta - \theta_0\right)}$$

$$\lambda = \frac{-c}{a \cos \theta + b \sin \theta}$$

where,

$$a = -c\beta^{-1} \cos \theta_0$$

$$b = c\beta^{-1} \sin \theta_0$$

and substituting $\tan \phi$ for λ

$$\tan \phi = \frac{-c}{a \cos \theta + b \sin \theta} \quad .$$

NOTE: This equation relates ϕ and θ explicity, ie, $\phi = f(\theta)$ or $f^{-1}(\phi) = \theta$.

Continuing with:

$$a \sin \phi \cos \theta + b \sin \phi \sin \theta = -c \cos \phi \quad .$$

Multiplying this through by the term ρ and rearranging gives:

$$a[\rho \sin \phi \cos \theta] + b[\rho \sin \phi \sin \theta] + c[\rho \cos \phi] = 0$$

Now, recognizing the relationship between spherical and Cartesian coordinates:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi \quad .$$

This constraint equation, in Cartesian coordinates, becomes:

$$ax + by + cz = 0 \quad .$$

This is a plane confined to pass through the end points (boundary conditions) and the sphere origin. The solution for set boundary conditions is unique <u>provided the end points are noncolinear with the origin.</u>

NOTE: $x = y = z = 0$, always a solution to the constraint plane.

Since the solution is confined to a plane (ie, linear) and the curve must be a GREAT circle (ie, includes the system origin) the distance between end points, s, can be written:

$$s = a' \Delta \quad ,$$

where Δ is the angle between the origin and vectors which point to the end points. These vectors can be called:

$$\vec{p}_0 = a' \left[\sin \phi_0 \cos \theta_0 \hat{i} + \sin \phi_0 \sin \theta_0 \hat{j} + \cos \phi_0 \hat{k}\right]$$

and,

$$\vec{p}_1 = a' \left[\sin \phi_1 \cos \theta_1 \hat{i} + \sin \phi_1 \sin \theta_1 \hat{j} + \cos \phi_1 \hat{k}\right] \quad .$$

Note,

$$|\vec{p}_0| = |\vec{p}_1| = a' \quad ,$$

and from linear algebra

$$\cos \Delta = \frac{\vec{p}_0 \cdot \vec{p}_1}{|\vec{p}_0|\,|\vec{p}_1|} \quad ,$$

therefore:

$$\cos \Delta = \sin \phi_0 \sin \phi_1 \cos \theta_0 \cos \theta_1 + \sin \phi_0 \sin \theta_0 \sin\phi_1 \sin \theta_1$$
$$+ \cos \phi_0 \cos \phi_1$$

$$\cos \Delta = \cos \phi_0 \cos \phi_1 + \sin \phi_0 \sin \phi_1 \left[\cos \theta_0 \cos \theta_1 + \sin \theta_0 \sin \theta_1\right]$$

$$\cos \Delta = \cos \phi_0 \cos \phi_1 + \sin \phi_0 \sin \phi_1 \cos \left(\theta_0 - \theta_1\right) .$$

This is exactly what Naperian spherical trigonometry (law of cosines) tells the reader to expect. The spherical law of sines can be recovered via geometrical construction.

If a boundary value problem is solved in such a way that a minima curve goes through two points, $\vec{p}_0$ and $\vec{p}_1$, then the minima solution is a plane of the form

$$\vec{N} \cdot \vec{x} = 0$$

where,

$$\vec{N} = (a\ b\ c) \quad ,$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

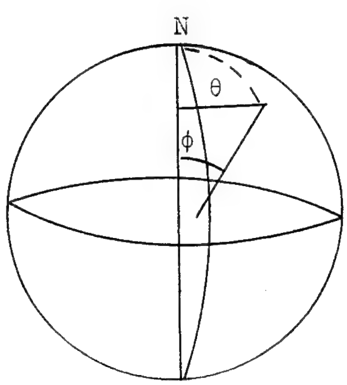
$\vec{N}$ is the planar normal vector (plane of the great circle) of arbitrary magnitude. Then given $\vec{p}_0$, $\vec{p}$, $\vec{N}$ can always be constructed (of arbitrary magnitude) as $\vec{N} = \vec{p}_0 \times \vec{p}_1$. If $\vec{N} \equiv 0$ when $\vec{p}_1 = \pm \vec{p}_0$, the solution is nonunique. In this case, there are infinite solutions which can be obtained by finding $\vec{N}$ such that either

$$\vec{p}_1 \cdot \vec{N} = 0 \quad \text{or, equivalently}$$

$$\vec{p}_1 \cdot \vec{N} = 0 \quad .$$

(1) Spherical detail on $\vec{H}$, the heading vector. Define p_0 and p_1 as points on a given minima curve. On the sphere, prerotate the coordinate system such that p_0 is in the same plane as the zero longitude and refer to p_1 in terms of $\left(\phi_1\ ,\ \Delta\theta\right)$.

A review of spherical coordinates.



With p_0 at $\theta = 0$, and using unit vectors:

$$\vec{p}_0 = \frac{1}{a}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\frac{\vec{p}_0}{|p_0|} = \sin\phi_0\hat{i} + \cos\phi_0\hat{k}$$

and for arbitrary p,

$$\frac{\vec{p}_1}{|p_1|} = \sin\phi_1 \cos\Delta\theta\hat{i} + \sin\phi_1 \sin\Delta\theta\hat{j} + \cos\phi_1\hat{k} \quad .$$

Now,

$$\frac{\vec{p}_0}{|\vec{p}_0|} \times \frac{\vec{p}_1}{|\vec{p}_1|} = \hat{i}\,(-\sin\phi_1 \sin\Delta\theta \cos\phi_0) + j\,(\sin\phi_1 \cos\Delta\theta \cos\phi_0$$

$$- \sin\phi_0 \cos\phi_1) + \hat{k}\,(\sin\phi_0 \sin\phi_1 \sin\Delta\theta)$$

also,

$$\left.\frac{\vec{\nabla} f}{|\vec{\nabla} f|}\right|_{p_0} = \frac{\vec{p}_0}{|\vec{p}_0|}$$

since

$$\nabla f = \frac{2}{a^2}\,(x\hat{i} + y\hat{j} + z\hat{k})$$, f the spherical surface function,

and

$$\left|\frac{\vec{\nabla} f}{|\vec{\nabla} f|}\right|_{p_0} = \sin\phi_0\hat{i} + \cos\phi_0\hat{k}$$

Therefore:

$$\vec{H} = \vec{N} \times \nabla f^*$$

$$N = p_0 \times p_1$$

*By definition, also refer to geometrical interpretation, since $\vec{N} = \vec{\nabla}\zeta$, the minima surface, (ie, plane).

and

$$\vec{H}_r = \begin{pmatrix} \sin\phi_1 \cos\Delta\theta \cos^2\phi_0 - \cos\phi_1 \sin\phi_0 \cos\phi_0 \\ \sin\phi_1 \sin\Delta\theta \\ \cos\phi_1 \sin^2\phi_0 - \sin\phi_1 \cos\Delta\theta \sin\phi_0 \cos\phi_0 \end{pmatrix}.$$

This vector now needs to be rotated into topocentric (SEZ)** coordinates. This is done by:

$$\vec{H}_{SEZ} = \begin{pmatrix} \cos\phi_0 & 0 & -\sin\phi_0 \\ 0 & 1 & 0 \\ \sin\phi_0 & 0 & \cos\phi_0 \end{pmatrix} H_r$$

calling,

$c = \cos\phi_0$

$s = \sin\phi_0$

$x = \sin\phi_1 \cos\Delta\theta$

$y = \sin\phi_1 \sin\Delta\theta$

$z = \cos\phi_1$

$$\vec{H}_{SEZ} = \begin{pmatrix} xc^3 - zsc^2 - zs^3 + xs^2c \\ y \\ xsc^2 - zs^2c + zcs^2 - xc^2s \end{pmatrix}$$

$$= \begin{pmatrix} c^2(xc - zs) + s^2(xc - zs) \\ y \\ 0 \end{pmatrix}$$

**SEZ, a local right-handed coordinate system which points south, east and up (or Z).

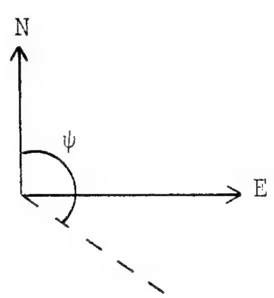
$$= \begin{pmatrix} xc - zs \\ y \\ 0 \end{pmatrix}.$$

and, finally, translating $\vec{H}_{SEZ}$ to $\vec{H}_{NEZ}$ (using north as a cardinal point and substituting)

$$H_{NEZ} = \begin{pmatrix} \cos \phi_1 \sin \phi_0 - \sin \phi_1 \cos \phi_0 \cos \Delta\theta \\ \sin \phi_1 \sin \Delta\theta \\ 0 \end{pmatrix}.$$

NOTE: The z component is zero, this must always be true in any case.

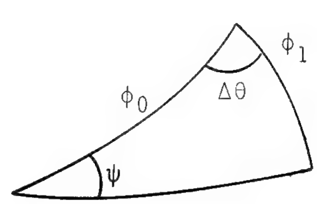
Now looking at the local system:



$$\tan \psi = \frac{E}{N} \tag{9}$$

$$\tan \psi = \frac{\sin \phi_1 \sin \Delta\theta}{\cos \phi_1 \sin \phi_0 - \sin \phi_1 \cos \phi_0 \cos \Delta\theta} \quad . \tag{10}$$

By leaving the result as a fraction, ie, $\tan \psi = \frac{E}{N}$, all quadrants are readily resolved. In an absolute sense, equation 10 could have just as easily been obtained via spherical trigonometry:



$$\cos c = \cos\phi_0 \cos\phi_1 + \sin\phi_0 \sin\phi_1 \cos\Delta\theta \tag{11}$$

$$\cos\phi_1 = \cos\phi_0 \cos c + \sin\phi_0 \sin c \cos\psi \tag{12}$$

$$\frac{\sin\psi}{\sin\phi_1} = \frac{\sin\Delta\theta}{\sin c} \tag{13}$$

continuing:

$$\sin c \sin\psi = \sin\phi \sin\Delta\theta \tag{14}$$

and from equation (12)

$$\sin c \cos\psi = \frac{1}{\sin\phi_0}\left[\cos\phi_1 - \cos\phi_0 \cos c\right]$$

and substituting from equation (11)

$$\sin c \cos\psi = \frac{1}{\sin\phi_0}\left[\cos\phi_1 - \cos\phi_0\left(\cos\phi_0 \cos\phi_1 + \sin\phi_0 \sin\phi_1 \cos\Delta\theta\right)\right]$$

$$= \frac{1}{\sin\phi_0}\left[\cos\phi_1\left(1 - \cos^2\phi_0\right) - \cos\phi_0 \sin\phi_1 \cos\Delta\theta\left(\sin\phi_0\right)\right]$$

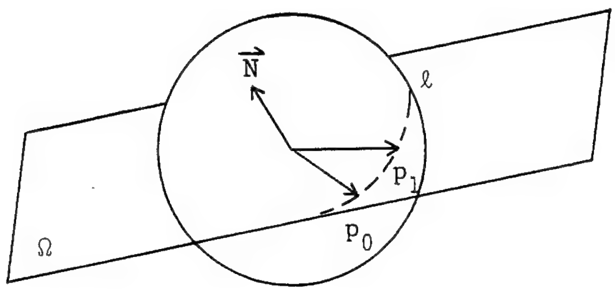
$$= \frac{1}{\sin\phi_0}\left[\sin\phi_0\left(\cos\phi_1 \sin\phi_0\right) - \sin\phi_0\left(\sin\phi_1 \cos\phi_0 \cos\Delta\theta\right)\right]$$

$$\sin c \cos\psi = \cos\phi_1 \sin\phi_0 - \sin\phi_1 \cos\phi_0 \cos\Delta\theta \; . \tag{15}$$

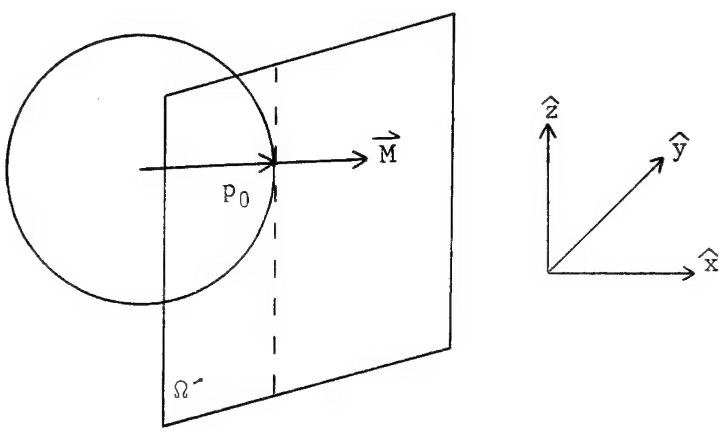
Dividing equation (14) by equation (15) gives:

$$\frac{\sin c \sin\psi}{\sin c \cos\psi} \equiv \tan\psi = \frac{\sin\phi_1 \sin\Delta\theta}{\cos\phi_1 \sin\phi_0 - \sin\phi_1 \cos\phi_0 \cos\Delta\theta} \; .$$

The geometrical interpretation of $\vec{H} = \vec{N} \times \vec{\nabla} f$ is determined from the following illustrations:



In this illustration curve ℓ represents the intersection of plane Ω and the spherical surface. Clearly $\vec{H}$ must reside in Ω, therefore be normal to $\vec{N}$ (ie, $\vec{N} \cdot \vec{H} = 0$ always).



From this illustration, Ω' is the planar surface normal to the point p. $\vec{H}$ must also reside in this plane. Headings (ie, north, south, east and west) are locally defined in this plane. $\vec{H}$ must also be normal to $\vec{M}$($\vec{M} \cdot \vec{H} = 0$ always).

NOTE: $\vec{N}$ is the surface normal for plane Ω, $\vec{M}$ is the surface normal for plane Ω').

$\vec{H}$ could be defined as the intersection of Ω and Ω' or:

$\vec{H} = \Omega \cap \Omega'$

Since the magnitude of $\vec{H}$ is not a factor, just the direction, then the fact that $\vec{N} \cdot \vec{H} = 0$ and $\vec{M} \cdot \vec{H} = 0$ can be used. From vector analysis a convenient way of finding a vector $\vec{A}$ which is normal to both $\vec{C}$ and $\vec{D}$ is:

$$\vec{A} = \vec{C} \times \vec{D}$$

then define $\vec{H}$,

$$\vec{H} = \vec{N} \times \vec{M} \quad .$$

Now $\vec{N}$ is obtained from $\vec{p}_0$ and $\vec{p}_1$,

$$\vec{N} = \vec{p}_0 \times \vec{p}_1 \quad ,$$

and $\vec{M}$, the surface normal vector is

$$\vec{M} = \vec{\nabla} f \quad ,$$

where,

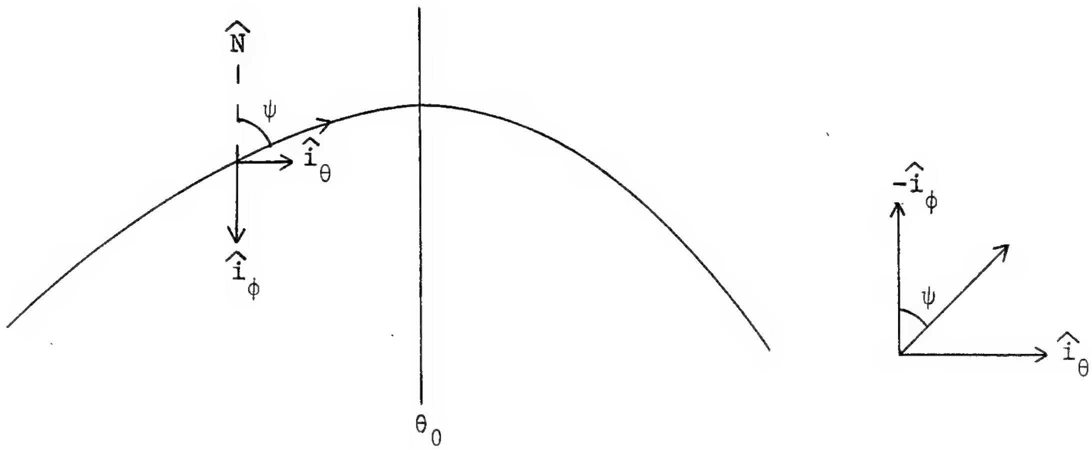
$f = f(x, y, z)$ which defines the surface.

Hence,

$$\vec{H} = \left(\vec{p}_0 \times \vec{p}_1 \right) \times \vec{\nabla} f \qquad \text{(for a sphere).}$$

$\vec{\nabla} f$ to be evaluated at a point in question.

The space curve directional vector $\vec{D}$ is found as follows:



For first quadrant direct:

$\rho = a$

$\phi = \tan^{-1} \lambda$

$\theta = \theta_0 - \sec^{-1} \frac{\lambda}{\beta}$ (λ decreasing, hence $-1 \cdot d\lambda$).

$$\vec{D}(\lambda) = 0\hat{i}_\rho + \frac{1}{\left(1 + \lambda^2\right)}(-1)\hat{i}_\phi + (-1)\frac{\lambda}{\left(1 + \lambda^2\right)^{1/2}} \cdot \frac{1}{\lambda\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2}}(-1)\hat{i}_\theta \; .$$

Measure ψ from true topographic north (ie, $-\hat{i}_\phi$ direction)

$$\vec{D}(\lambda) = \frac{1}{\left(1 + \lambda^2\right)}\left(-\hat{i}_\phi\right) + \frac{1}{\left(1 + \lambda^2\right)^{1/2}\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2}} \hat{i}_\theta$$

$$\vec{D} \cdot \hat{i}_\phi = \cos \psi |\vec{D}|$$

$$\vec{D} \cdot \hat{i}_\theta = \sin \psi |\vec{D}|$$

$$\tan \psi = \frac{\left(1 + \lambda^2\right)^{1/2}}{\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2}} \; .$$

NOTE: From this $\cot\psi = \cos\phi \tan\left(\theta_0 - \theta\right)$.

$$\psi = \tan^{-1}\left(\beta\left(\frac{\lambda^2 + 1}{\lambda^2 - \beta^2}\right)^{1/2}\right).$$

For $\vec{H}$,

$$\vec{H} = \vec{\nabla}\zeta \times \vec{\nabla}S$$

Now, from the surface constraint:

$$\rho - a = 0 \quad ,$$

and for the minimum constraint:

$\beta \sec\left(\theta_0 - \theta\right) - \tan\phi = 0$ (since in the $\vec{D}$ derivation, a quadrant in which $\theta < \theta_0$ was used, thus making the argument positive).

$$\vec{\nabla}S = \hat{i}\rho$$

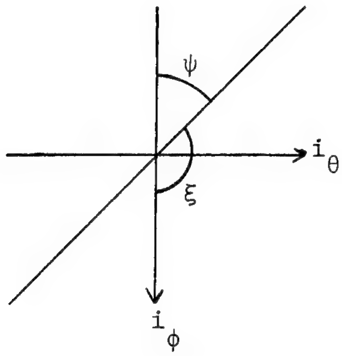
$$\vec{\nabla}\zeta = \frac{-1}{\rho}\sec^2\phi\ \hat{i}_\phi + \frac{-1}{\rho\sin\phi}(\beta\sec\Delta\theta\tan\Delta\theta)\ \hat{i}_\theta \Big| \left[\text{evaluated at } \rho\left(\phi\ ,\ \Delta\theta\right)\right] .$$

Then,

$$\vec{H} = \begin{pmatrix} i_\rho & i_\phi & i_\theta \\ 0 & -\dfrac{\sec^2\phi}{\rho} & -\dfrac{\beta\sec\Delta\theta\tan\Delta\theta}{\rho\sin\phi} \\ 1 & 0 & 0 \end{pmatrix}$$

$$\vec{H} = 0i_\rho - \frac{\beta\sec\Delta\theta\tan\Delta\theta}{\rho\sin\phi}\hat{i}_\phi + \frac{\sec^2\phi}{\rho}\hat{i}_\theta .$$

Now, in the local system



$$\frac{\vec{H}}{|\vec{H}|} \cdot \hat{i}_\phi = \cos \xi = -\cos \psi$$

$$\frac{\vec{H}}{|\vec{H}|} \cdot \hat{i}_\theta = \sin \xi = \sin \psi$$

therefore,

$$\frac{\vec{H} \cdot \hat{i}_\phi}{H \cdot \hat{i}_\theta} = -\cot \psi = \frac{-\dfrac{\beta \sec \Delta\theta \tan \Delta\theta}{\rho \sin \phi}}{\dfrac{\sec^2 \phi}{\rho}}$$

and,

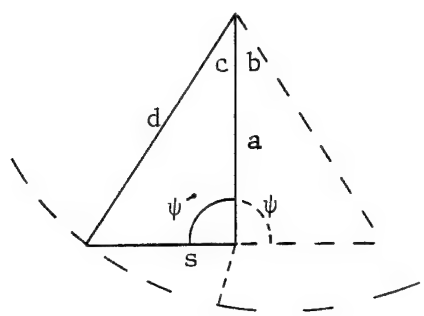
$$\cot \psi = \frac{\beta \sec \Delta\theta}{\tan \phi} \cos \phi \tan \Delta\theta$$

but,

$$\beta \sec \Delta\theta = \tan \phi$$

$$\cot \psi = \cos \phi \tan \Delta\theta \quad \text{QED.}$$

Finally, the relation for ψ could have been derived directly from spherical trigonometry.



given b, a , find ψ.

b, a implies a, c, d, ψ'

$\psi = \pi - \psi'$,

therefore,

$\tan \psi = \tan \psi'$

with

$d = \frac{\pi}{2}$

and

$c + b = \frac{\pi}{2}$

$\sin \psi = \frac{\sin c}{\sin s}$

$\cos d = 0 = \cos a \cos s + \sin a \sin s \cos \psi'$

$-\cos a \cos s = \sin a \sin s \cos \psi'$

$-\cos \psi' = \frac{\cos a \cos s}{\sin a \sin s}$

$$-\tan\psi' = \frac{\dfrac{\sin c}{\sin s}}{\dfrac{\cos a \cos s}{\sin a \sin s}} = \frac{\sin c}{\cos s} \tan a = \frac{\sin c}{\sin a \cos c} \cdot \frac{\sin a}{\cos a} = \frac{\tan c}{\cos a}$$

since

$$\cos s = \sin a \cos c \quad .$$

Continuing with

$$c + b = \frac{\pi}{2} \quad ,$$

$$\tan c = \cot b$$

$$\tan \psi = \frac{\cot b}{\cos a}$$

$$\cot \psi = \cos a \tan b$$

but

$$a = \phi$$

$$b = \Delta\theta$$

$$\cot \psi = \cos \phi \tan \Delta\theta$$

or

$$\tan \psi = \frac{\sec \phi}{\tan \Delta\theta}$$

NOTE: $\cos s = \cos a \cos d + \sin a \sin d \cos c$, $\cos s = \sin a \cos c$ for $d = \frac{\pi}{2}$.

For distance measurement:

$$\Delta s = a \int \frac{1}{1 + \lambda^2} \left[1 + \frac{1 + \lambda^2}{\dfrac{\lambda^2}{\beta^2} - 1} \right]^{1/2} d\lambda$$

$$\Delta s = a\int \frac{1}{1+\lambda^2}\cdot\frac{1}{\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}}\left[\frac{\lambda^2}{\beta^2}-1+1+\lambda^2\right]^{1/2} d\lambda$$

$$\Delta s = a\int \frac{1}{1+\lambda^2}\cdot\frac{1}{\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}}\lambda d\lambda\left[\frac{1}{\beta^2}+1\right]^{1/2}$$

$$\Delta s = a\left(1+\beta^2\right)^{1/2}\int\frac{1}{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}\lambda d\lambda \quad .$$

Substituting:

$1 + \lambda^2 = u$

$2\lambda d\lambda = du$

$\lambda^2 = u - 1$

$$\Delta s = \frac{a\left(1+\beta^2\right)^{1/2}}{2}\int\frac{du}{u\left(u-\alpha\right)^{1/2}} \qquad \left[\alpha = \left(\beta^2+1\right)\right] \quad .$$

Substituting back for u

$$\Delta s = a\frac{\left(1+\beta^2\right)^{1/2}}{2}\cdot\frac{2}{\left(\beta^2+1\right)^{1/2}}\tan^{-1}\left|\frac{u-\left(\beta^2+1\right)}{\beta^2+1}\right|^{1/2}\left(\left[\frac{1+\lambda^2-\beta^2-1}{\beta^2+1}\right]^{1/2}\right)$$

$$\Delta s = a\tan^{-1}\left[\frac{\lambda^2-\beta^2}{1+\beta^2}\right]^{1/2}\Bigg|_{\lambda_0}^{\lambda_1} \quad .$$

The five equation solution for the sphere:

$\rho = a$

$\phi = \tan^{-1}\lambda$

$\theta = \theta_0 \pm \sec^{-1}\frac{\lambda}{\beta}$

$$\Delta s(\lambda) = a\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\Bigg|_{\lambda_0}^{\lambda_1}$$

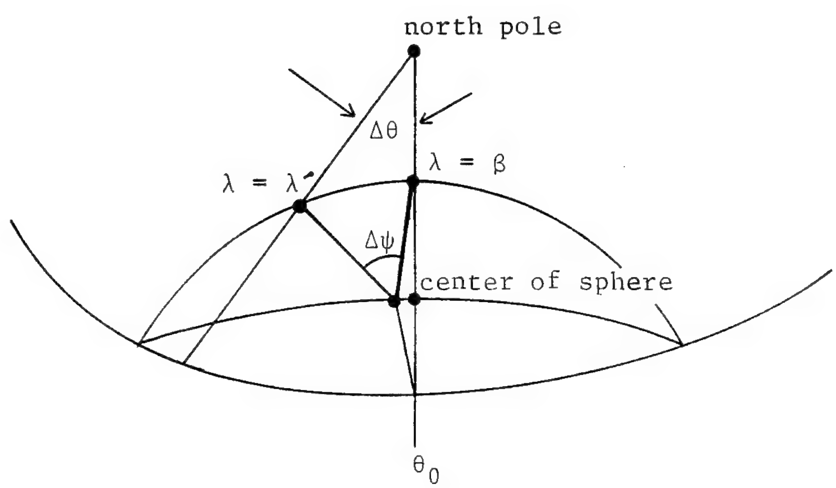
$$\psi(\lambda) = \tan^{-1}\left[\beta\left(\frac{\lambda^2 + 1}{\lambda^2 - \beta^2}\right)^{1/2}\right] .$$

From the five equations for the sphere, combine the equations for Δs and θ, eliminating the parametric variable λ.

$$\theta = \theta_0 \pm \sec^{-1}\frac{\lambda}{\beta}$$

$$\Delta s = a \tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2} \Bigg|_{\lambda_0}^{\lambda_1} .$$

From the spherical geodesic (great circle route) $\frac{\Delta s}{a} = \Delta\bar{\psi}$, the central swept out angle, evaluate from the top of the curve down one side to $\lambda = \lambda'$.



$$\tan\frac{\Delta s}{a} = \left(\frac{\lambda'^2 - \beta^2}{1 + \beta^2}\right)^{1/2}$$

$$|\Delta\theta| = |\theta - \theta_0| = \sec^{-1}\frac{\lambda'}{\beta}$$

hence,

$$\lambda' = \beta \sec \Delta\theta$$

$$\frac{\Delta s}{a} = \Delta\bar{\psi}$$

and

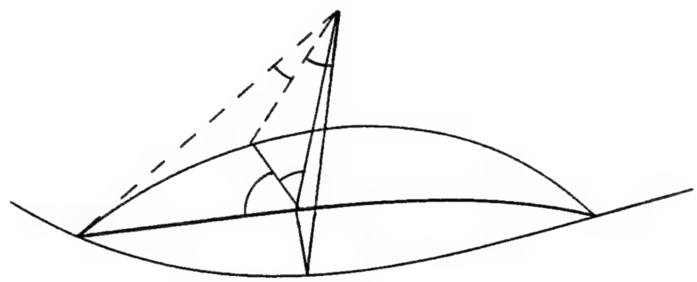
$$\tan \Delta\bar{\psi} = \frac{\beta}{\left(1 + \beta^2\right)^{1/2}} \left(\sec^2 \Delta\theta - 1\right)^{1/2}$$

$$= \frac{\beta}{\left(1 + \beta^2\right)^{1/2}} \tan \Delta\theta \quad .$$

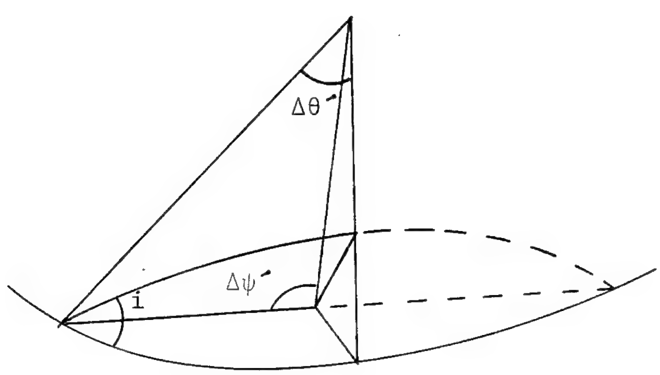
Now, let

$$\Delta\theta' = \frac{\pi}{2} - \Delta\theta$$

$$\Delta\bar{\psi}' = \frac{\pi}{2} - \Delta\bar{\psi}$$



So that instead of measuring the swept out angle from the top of the curve, the measurement is taken from the nodal crossing.



Note that for a given geodesic β is related to the minimum colatitude value that geodesic curve will attain. If $\phi_{min} = \hat{\phi}$, then $\beta = \tan \hat{\phi}$. $\hat{\phi}$ is related to the inclination of the geodesic curve (remember, the curve is coplanar) by

$$i = \frac{\pi}{2} - \hat{\phi}$$

hence,

$$\beta = \cot i \quad ,$$

$$\frac{\beta}{\left(1 + \beta^2\right)^{1/2}} = \cos i$$

and

$$\tan\left(\frac{\pi}{2} - \Delta\bar{\psi}'\right) = \cos i \tan\left(\frac{\pi}{2} - \Delta\theta'\right)$$

$$\cot \Delta\bar{\psi}' = \cos i \cot \Delta\theta'$$

$$\tan \Delta\bar{\psi}' \cdot \cos i = \tan \Delta\theta' \quad .$$

This is completely consistent with Napier's findings and is sometimes referred to as the spherical law of tangents.

Although most of the attention has been northern hemisphere oriented, the solution, by symmetry, is equally valid for southern hemisphere trajectories $(-\infty \leq \lambda \leq 0)$. Singularities occur at $\lambda = 0$ (either north or south pole) and at $\lambda \to \infty$ at the equator. From a computational point of view, equatorial, near equatorial and transequatorial routes $(\lambda \to \infty)$ are inconvenient to calculate. Defining the variable $x = \frac{1}{\lambda}$ and instead of $\lambda \to \infty$, let $x \to 0$. For this case, the formulas become:

$$\rho = a$$

$$\phi(x) = \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\theta(x) = \theta_0 \pm \cos^{-1} (\beta x)$$

$$\Delta s(x) = a\left[\frac{\pi}{2} - \tan^{-1}\left[x\left(\frac{1 + \beta^2}{1 - \beta^2 x^2}\right)^{1/2}\right]\right] \qquad \left(x \leq \frac{1}{\beta}\right)$$

$$\psi(x) = \tan^{-1} \beta\left(\frac{1 + x^2}{1 - \beta^2 x^2}\right)^{1/2} \quad .$$

For geodesics which cross the equator note that $\tan \psi = \beta$ at the equator. The geodesics are assumed to be smooth, continuous curves so that for a given β in the northern hemisphere, the southern hemisphere continuation of the same geodesic will have the save value of β in the absolute sense, thus both a given geodesic curve and its slope are continuous across the equatorial boundary.

Other singularities of interest are:

$\lim \beta \to 0$.

This becomes a transpolar trajectory:

$\theta = \theta_0 - \frac{\pi}{2}$ on one side

$\theta = \theta_0 + \frac{\pi}{2}$ on the other. (ie, $\theta(\lambda)$ becomes a discontinuous step function, becoming undefined at $\lambda = 0$, at either pole. Remember $\lambda \geqslant \beta$ always).

$\lim \beta \to \infty$. This becomes an equatorial trajectory: $\phi \equiv \frac{\pi}{2}$. In that $\lambda \geqslant \beta$ this becomes computationally difficult to handle. The matrix solution removes this singularity.

$\lim \lambda \to \beta$.

$$\psi(\lambda = \beta) = \tan^{-1} \beta \left(\frac{\lambda^2 + 1}{\lambda^2 - \beta} \right)^{1/2} .$$

This can be rewritten as

$$\psi(\lambda) = \frac{\pi}{2} - \tan^{-1} \beta \left(\frac{\lambda^2 - \beta^2}{\lambda^2 + 1} \right)^{1/2} .$$

Thus, as $\lambda \to \beta$,

$$\psi(\lambda = \beta) = \frac{\pi}{2} .$$

For a typical boundary value problem, given $\hat{\rho}_0$, $\hat{\rho}_1$, find the geodesic curve which contains $\vec{\rho}_0$, $\vec{\rho}_1$. The expression $\vec{N} = \vec{\rho}_0 \times \vec{\rho}_1$ is convenient to use. In the case of physical interest, where $\vec{N} \equiv 0$, $\vec{\rho}_0 = -\vec{\rho}_1$ there are an infinite number of geodesics which will satisfy the condition. (This is not generally true in the slightly oblate case, as will be seen in IVb(2).

(2) Existence and uniqueness of solutions. One can use the idea of breaking up the surface into small, differential grids, and numerically find a curve between end points through the patchwork of grids which will minimize the overall distance; but, in general, there is no guarantee that this curve is unique. Neither for the curve $\hat{f}$, nor $\hat{f}'$ (as defined in the closest distance planar section). This is most dramatically seen on the sphere. Although there are an infinite number of cases, only two are cited.

(a) Start point north pole, end point south pole. The minima curve exists, but is not unique. Take any meridian to the south pole ($\hat{f}$ exists, not unique).

(b) Point ρ at the north pole, the object is to get to the equator. The intercept curve is any meridian to the equator (f' exists and satisfies all conditions, but is not unique).

Uniqueness is highly surface and end point dependent. Solutions on the sphere show a parallel of sorts with the general solutions of a second order differential equation as they should.

Given a point and a slope, the minima curve is uniquely determined. Given end points, the minima curve may not be unique and there may exist infinite solutions.

Looking at the spherical case in more detail, two cases (or subcases of particular significant physical interest) are identified.

Case 1, given $\vec{\rho}_0$ and ψ*, determine the minima curve parameters.

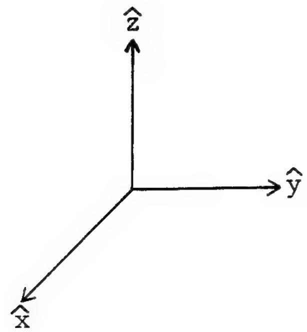
Case 2, given $\vec{\rho}_0$ and $\vec{\rho}_1$, determine the minima curve parameters.

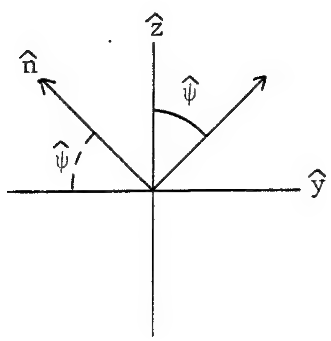
In each case, assume $\vec{\rho}_0$ is at $\theta = 0$. (If not, use relative θ coordinates. Remember, the surface is one of revolution and symmetry can be used.)

And in each case it is most convenient to work with the solution planar normal $\vec{N}$, which in case 2 is easily determined via $\vec{N} = \vec{\rho}_0 \times \vec{\rho}_1$.

Now, consider the relation of $\vec{N}$ with ψ :

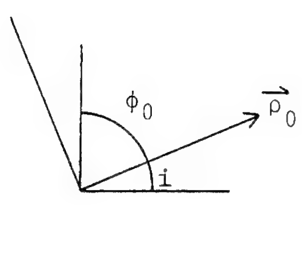
*ψ, a topocentrically defined heading angle ($0 \leq \psi \leq 360$).

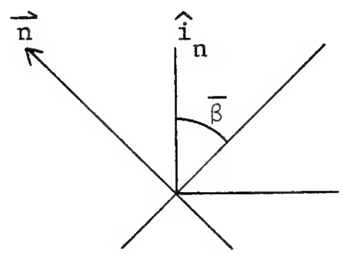




$$\vec{N} = 0\hat{i}_x - \cos\psi\hat{i}_y + \sin\psi\hat{i}_z$$

$\hat{z}$, $\hat{y}$ a topocentric system on the surface.





NOTE: $\sin \vec{\beta} = \vec{N} \cdot \hat{i}_K$

Rotate the normal up to the point $\vec{\rho}_0$ (at ϕ_0) this is a $\hat{y}$ axis rotation through an angle $\frac{\pi}{2} - \phi_0$ so that

$$\vec{N} = \begin{pmatrix} \sin\phi_0 & 0 & -\cos\phi_0 \\ 0 & 1 & 0 \\ \cos\phi_0 & 0 & \sin\phi_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\cos\psi \\ \sin\psi \end{pmatrix}$$

$$\vec{N}(\phi_0) = \begin{pmatrix} -\cos\phi_0 \ \sin\psi \\ -\cos\psi \\ \sin\phi_0 \ \sin\psi \end{pmatrix},$$

which must then be rotated $(-\theta)$ about the z axis for proper inertial (I,J,K) orientation (θ is defined here as the longitude of $\vec{\rho}_0$).

This equation relates the two cases, as given $\vec{N}$, which is equal to $\vec{\rho}_0 \times \vec{\rho}_1$ and forcing $|\vec{N}| = 1$, then ψ at ϕ_0 is determined via the above relation, and given $\vec{\rho}_0$, ψ then $\vec{N}$ is obtained (remember to keep track of θ, which was assumed to be zero).

Now the angle $\bar{\beta}$ is easily determined via $\sin \bar{\beta} = \vec{N} \cdot \hat{i}_k$ for either case (reference illustration)*.

NOTE: θ_0 the spherical/sphere case parameter is obtained from $\vec{\theta}_{node} = \vec{k} \times \vec{N}$, from which (depending on quadrant) $\theta_0 = \frac{\pi}{2} + \tan^{-1} \dfrac{\vec{\theta} \cdot \hat{j}}{\vec{\theta} \cdot \hat{i}}$. By geometry $\vec{\theta}_{node}$ points along the equator to where the plane defined by $\vec{N}$ intersects the sphere equatorial plane.

The angle $\bar{\beta}$ is related to the spherical/sphere case parameter β via $\beta = \tan \bar{\beta}$.

Given a curve (θ_0, β) and some arbitrary point $\vec{\rho}_a$, determine if the closest point on the curve to point $\vec{\rho}_a$ can be found.

From,

$$\tan \phi = \beta \sec (\theta - \theta_0)$$

obtain the curve in the cartesian form

$$ax + by + cz = 0$$

$$(\vec{N} \cdot \vec{x} = 0)$$

and let,

$$\vec{P}_a = x_a\hat{i} + y_a\hat{j} + z_a\hat{k} \quad .$$

Now, note if

$$ax_a + by_a + cz_a = 0$$

then $\vec{P}_a$ is on the curve. If this is so, stop the process. If not, continue.

*Require $|\vec{N}| \equiv 1$.

Let $|\vec{N}| = 1$, $|\vec{P}_a| = 1$*

construct $\vec{A}$, $\vec{B}$ such that

$$\vec{A} \cdot \vec{B} = 0$$

$$\vec{A} + \vec{B} = \vec{P}_a$$

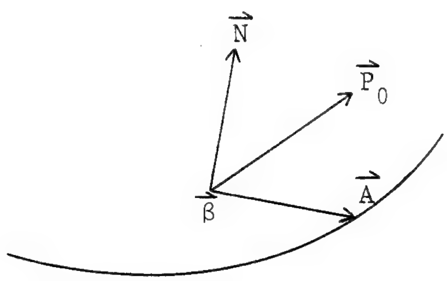
$$\frac{\vec{A}}{|\vec{A}|} = \vec{N} \ .$$

So:

$$\vec{A} = |\vec{P}_a \cdot \vec{N}|\vec{N}$$

$$\vec{B} = \vec{P}_a - \vec{A} \ .$$

By geometry $\vec{B}$ points to the solution point with



$$\vec{B} = \vec{P}_a - |\vec{P}_a \cdot \vec{N}|\vec{N} \ ,$$

then let

$$|\vec{B}| = 1$$

and from this $\vec{B}$ the solution is extracted.

*where $\vec{N} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{\rho}_a = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$

NOTE: If $\vec{P}_a$ is in the direction of $\vec{N}$, $\vec{B} \equiv 0$, there is no unique solution. Also if $\vec{P}_a \cdot \vec{N} = 0$, $\vec{A} = 0$, $\vec{B} = \vec{P}_a$, the point is on the curve (this is an application of Gram-Schmidt orthogonalization).

(3) Solution via matrix methods.

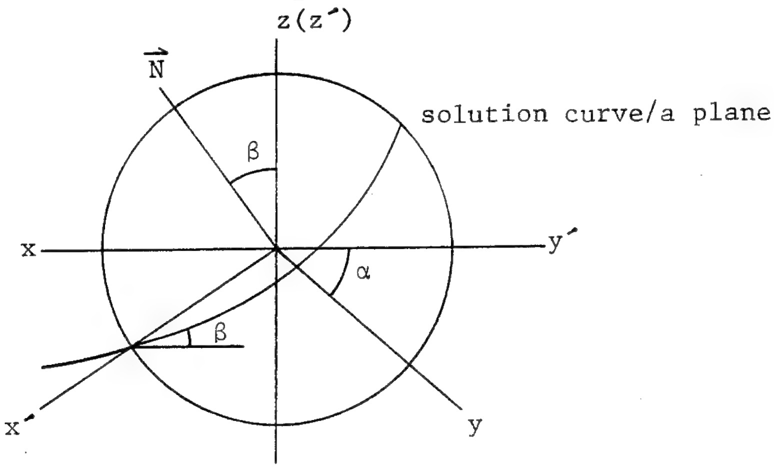
(a) Development. In that, if $\vec{P}(t)$ is the position vector of the solution of the spherical case (say the general case, where $P(\vec{t})$ is a vector function of time measured from the origin) it is easily shown that for a geodesic:

1. $\vec{P}(t) \cdot \vec{P}(t) = a^2$, a constant for any time.

2. There exists a vector $\vec{N}$ such that $\vec{N} \cdot \vec{P}(t) = 0$, for all t.

The two results indicate a solution obtainable in terms of vector matricies.

Consider the following diagram.

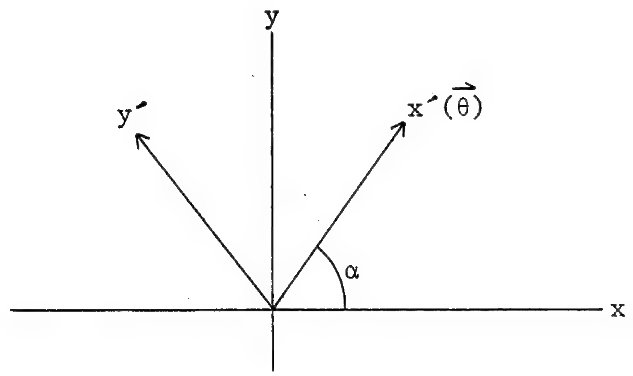


Lets say $\vec{N}$ is known in the x, y, z systems and $|\vec{N}| = 1$

$\vec{N} \cdot \hat{k} = \cos \beta$

$-\vec{N} \times \hat{k} = \vec{\theta}$ (which points in the direction of $x´$).

Now



Let $\vec{\theta}$ have components $a\hat{i} + b\hat{j}$

hence

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \tan^{-1}\left(\frac{\vec{\theta} \cdot \hat{j}}{\vec{\theta} \cdot \hat{i}}\right) \qquad \text{(all quadrants resolved).}$$

The known vector $\vec{N}$ contains the two necessary components needed to exactly specify the curve (of minimum distance).

From $\vec{P}(t) \cdot \vec{P}(t)$ = constant assume a solution in the $\vec{N} \cdot \vec{x}$ plane of the form:

$$\vec{P}(t) = \begin{pmatrix} \cos \bar{\psi} \\ \sin \bar{\psi} \\ 0 \end{pmatrix}$$

where, if the minimum distance trajectory is one of constant angular velocity, $\bar{\psi} = \omega t + \delta$, δ an offset angle. $\bar{\psi}$ the internal swept out angle. This vector referenced to the x', y', z' system (a rotation of β about the x axis) is:

$$\vec{P}'(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \psi \\ \sin \psi \\ 0 \end{pmatrix}$$

and finally referenced to the x, y, z system.

Let α, β be the rotation angles which take $\hat{\vec{P}}$ in the $\beta \equiv 0$ plane, to $\vec{P}$ the actual position vector.

Now,

$$\vec{P} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \left(\hat{\vec{\rho}} \right)$$

$$\vec{P} = \begin{pmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\sin\beta\cos\alpha \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\bar{\psi} \\ \sin\bar{\psi} \\ 0 \end{pmatrix}$$

$$\vec{N} = \begin{pmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\sin\beta\cos\alpha \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{p} = (\cos\alpha\cos\bar{\psi} - \sin\alpha\cos\beta\sin\bar{\psi})\hat{\imath} + (\sin\alpha\cos\bar{\psi} + \cos\alpha\cos\beta\sin\bar{\psi})\hat{\jmath} + (\sin\beta\sin\bar{\psi})\hat{k}$$

and

$$\Rightarrow \vec{N} = (\sin\alpha\sin\beta)\hat{\imath} - (\sin\beta\cos\alpha)\hat{\jmath} + (\cos\beta)\hat{k} \quad .$$

If $\vec{N}$ is given (from boundary or initial conditions) α, β are easily determined.

The objective now is to show:

$$\frac{d\vec{P}}{dt} = \vec{N} \times \vec{P} \quad .$$

NOTE: $|\vec{P}| = 1$, $|\vec{N} \times \vec{P}| = 1$, since $|\vec{N}| = 1$, $|\vec{P}| = 1$, $\vec{P} \cdot \vec{N} = 0$.

In order to keep

$$\left|\frac{d\vec{P}}{dt}\right| = 1$$

let,

$$\sin\bar{\psi} \rightarrow \cos\bar{\psi}$$

and

$\cos \bar{\psi} \rightarrow - \sin \bar{\psi}$

for the case of taking unit derivatives.

Then,

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} -\cos \alpha \sin \bar{\psi} - \sin \alpha \cos \beta \cos \bar{\psi} \\ \cos \alpha \cos \beta \cos \bar{\psi} - \sin \alpha \sin \bar{\psi} \\ \sin \beta \cos \bar{\psi} \end{pmatrix}$$

for $\vec{N} \times \vec{P}$

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ N_0 & N_1 & N_2 \\ P_0 & P_1 & P_2 \end{pmatrix}$$

$$\hat{i}(N_1 P_2 - P_1 N_2) + \hat{j}(P_0 N_2 - N_0 P_2) + \hat{k}(N_0 P_1 - P_0 N_1) \quad .$$

With:

$N_0 = \sin \alpha \sin \beta$

$N_1 = -\cos \alpha \sin \beta$

$N_2 = \cos \beta$

$P_0 = \cos \alpha \cos \bar{\psi} - \sin \alpha \cos \beta \sin \bar{\psi}$

$P_1 = \sin \alpha \cos \bar{\psi} + \cos \alpha \cos \beta \sin \bar{\psi}$

$P_2 = \sin \beta \sin \bar{\psi} \quad .$

Evaluating each component

$$(\vec{N} \times \vec{p}) \cdot \hat{i} = -\cos\alpha \sin\beta(\sin\beta \sin\bar{\psi}) - \cos\beta(\sin\alpha\cos\bar{\psi} + \cos\alpha\cos\beta\sin\bar{\psi})$$

$$= -\sin^2\beta \cos\alpha \sin\bar{\psi} - \cos\beta \sin\alpha \cos\bar{\psi} - \cos\alpha\cos^2\beta \sin\bar{\psi}$$

$$= -\cos\beta \sin\alpha \cos\bar{\psi} - \cos\alpha \sin\bar{\psi} \ .$$

$$(\vec{N} \times \vec{p}) \cdot \hat{j} = \cos\beta(\cos\alpha\cos\bar{\psi} - \sin\alpha\cos\beta\sin\bar{\psi}) - \sin\alpha\sin^2\beta\sin\bar{\psi}$$

$$= \cos\beta\cos\alpha\cos\bar{\psi} - \sin\alpha\cos^2\beta\sin\bar{\psi} - \sin\alpha\sin^2\beta\sin\bar{\psi}$$

$$= \cos\beta\cos\alpha\cos\bar{\psi} - \sin\alpha\sin\bar{\psi} \ .$$

$$(\vec{N} \times \vec{p}) \cdot \hat{k} = \sin\alpha\sin\beta(\sin\alpha\cos\bar{\psi} + \cos\alpha\cos\beta\sin\bar{\psi}) + \cos\alpha\sin\beta(\cos\alpha\cos\bar{\psi} - \sin\alpha\cos\beta\sin\bar{\psi})$$

$$= \sin^2\alpha\sin\beta\cos\bar{\psi} + \sin\alpha\cos\alpha\sin\beta\cos\beta\sin\bar{\psi} + \cos^2\alpha\sin\beta\cos\bar{\psi} - \cos\alpha\sin\alpha\sin\beta\cos\beta\sin\bar{\psi}$$

$$= \sin\beta\cos\bar{\psi}$$

hence,

$$\frac{d\vec{P}}{dt} = \vec{N} \times \vec{P} \qquad \text{(as expected).}$$

$$\frac{d\vec{P}}{dt} = \nabla\vec{\zeta} \times \vec{\nabla}s \ .$$

Since $\vec{N} = \vec{\nabla}\zeta$ and the surface gradient for a sphere $\vec{\nabla}S = \vec{P}$, what has been shown is that

$$\frac{d\vec{P}}{dt} = \vec{H} \quad \text{which is what would be intuitively expected.}$$

Continuing with,

$$\vec{P}(t) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\bar{\psi} \\ \sin\bar{\psi} \\ 0 \end{pmatrix}$$

$(\bar{\psi} = \omega(t) \pm \delta)$.

Finally, the term δ, t = 0 needs to be evaluated.

Lets say the term $\vec{P}(0)$ is known

then,

$$\vec{P}(0) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\alpha} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}_{\beta} \begin{pmatrix} \cos\delta \\ \sin\delta \\ 0 \end{pmatrix}$$

from which δ is easily solved.

NOTE: For the argument δ may be $\bar{\psi} = \omega(t) + \delta$ or $\bar{\psi} = \omega(t) - \delta$.

The $\pm$ determination can always be made from the fact that,

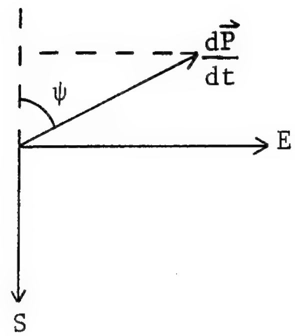
$$\cos\delta = \frac{\vec{P}_0 \cdot \vec{\theta}}{|\vec{P}_0||\vec{\theta}|}$$

$$\sin\delta = \frac{|\vec{P}_0 \times \vec{\theta}|}{|\vec{P}_0||\vec{\theta}|}$$

Direct trajectory can be defined for ω and retrograde trajectory for $-\omega$ as usual.

Now, look for heading information for $\vec{P}(t)$.

Given $\vec{P}(t)$ and with known factors $\{\phi, \theta\}$ or {latitude, longitude} data $\frac{d\vec{P}(t)}{dt}$ can be found. Then with the {latitude, longitude} data rotate this vector $\frac{d\vec{P}}{dt}$ into the topocentric SEZ system (the Z component of $\frac{d\vec{P}}{dt}$ must go to zero in this system) and obtain:



$$-\frac{d\vec{P}'}{dt}\cdot\hat{S} = \cos\psi$$

$$\frac{d\vec{P}'}{dt}\cdot\hat{E} = \sin\psi\ .$$

hence (where $\hat{S}, \hat{E}, \hat{Z} \rightarrow \hat{i}', \hat{j}', \hat{k}'$)

$$\tan\psi = \frac{\frac{d\vec{P}}{dt}\cdot\hat{E}}{-\frac{d\vec{P}}{dt}\cdot\hat{S}} = \frac{d\vec{P}'\cdot\hat{j}'}{-d\vec{P}'\cdot\hat{i}'}$$

$\vec{P}'$ is the position vector in the topocentric system, related $\vec{P}$ via

$$\vec{P}'(t) = \begin{pmatrix} \sin L & 0 & -\cos L \\ 0 & 1 & 0 \\ \cos L & 0 & \sin L \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{P}(t)\ .$$

NOTE: L and θ are obtained from $\vec{P}(t)$ as $\vec{P}(t) = \begin{pmatrix} \cos L\cos\theta \\ \cos L\sin\theta \\ \sin L \end{pmatrix}$.

Let,

$$\vec{P} = \sin\phi\cos\theta\hat{i} + \sin\phi\sin\theta\hat{j} + \cos\phi\hat{k}$$

and, noting $\phi + L = \frac{\pi}{2}$ (L, the latitude variable)

so that

$$\vec{P}_i = \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{pmatrix} \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

$$\vec{P}_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{k} \quad .$$

This is exactly as it should be since $\vec{P}_i \equiv \hat{k}$ as seen in the local topocentric system. Letting M be the topocentric rotation matrix,

$$\vec{P}_i = M\,\vec{P}_0$$

Now,

$$\vec{P}_0 = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}_{\phi,\theta} = \begin{pmatrix} \cos\alpha\cos\bar{\psi} - \sin\alpha\cos\beta\sin\bar{\psi} \\ \sin\alpha\cos\bar{\psi} + \cos\alpha\cos\beta\sin\bar{\psi} \\ \sin\beta\sin\bar{\psi} \end{pmatrix}_{\alpha,\beta,\bar{\psi}}$$

(this relates ϕ, θ to α, β, $\bar{\psi}$. $\bar{\psi}$ is the internal swept out angle).

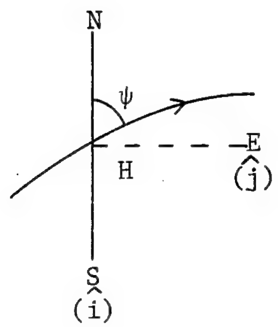
For the heading vector development at the point $\vec{P}$, let $\vec{H}_r = M\,\vec{H}$. $\vec{H}_r$ is then the heading vector in the topocentric system.

$$\vec{H}_r = M\,\frac{d\vec{P}_0}{dt} \quad ,$$

NOTE: $\vec{H} \cdot \hat{k} \equiv 0$, since at the surface point $\vec{H}_r$ is tangential to the surface.

ψ, the heading angle as measured in the local topocentric plane can be obtained as usual,

$$\psi = \tan^{-1}\left(- \frac{\vec{H}_r \cdot \hat{j}}{\vec{H}_r \cdot \hat{i}} \right) \text{ (note the negative sign.)}$$



(b) Summary of matrix solution for the spherical case.

Let $\vec{P}$ be the position vector:

$$\vec{P} = a\begin{pmatrix} \sin\phi\ \cos\theta \\ \sin\phi\ \sin\theta \\ \cos\phi \end{pmatrix}.$$

From the surface constraint

$$|\vec{P}| = a\ .$$

From the minimum distance constraint obtain

$$\vec{P}\cdot\vec{N} = 0\ .$$

Let $\begin{pmatrix} \cos\bar{\psi} \\ \sin\bar{\psi} \\ 0 \end{pmatrix}$ be the vector measured in the $\vec{N}\cdot\vec{X} = 0$ plane (central angle,

$\bar{\psi}$, measured from the equatorial node). Then θ_0, β initial/boundary parameters can be translated to α, γ angle constraints such that

$$\vec{P} = \begin{pmatrix} \cos\alpha & -\sin\alpha\ \cos\gamma & \sin\alpha\ \sin\gamma \\ \sin\alpha & \cos\alpha\ \cos\gamma & -\cos\alpha\ \sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}\begin{pmatrix} \cos\bar{\psi} \\ \sin\bar{\psi} \\ 0 \end{pmatrix}$$

θ_0, β determine $\vec{N}$ and,

$\vec{N} = \sin\alpha \sin\gamma\hat{i} - \cos\alpha \sin\gamma\hat{j} + \cos\gamma\hat{k}$.

$\bar{\psi}$ can be written:

$\bar{\psi}(t) = \bar{\psi}_0 + \frac{v}{a}\Delta t$ (for constant velocity trajectory).

$$\vec{H} = \begin{pmatrix} \cos\alpha & -\sin\alpha\cos\gamma & \sin\alpha\sin\gamma \\ \sin\alpha & \cos\alpha\cos\gamma & -\cos\alpha\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} \frac{d\vec{P}}{dt}$$ ($\vec{H}$, the directional vector).

$\bar{\psi}_0$ can be determined as

$$\cos\bar{\psi}_0 = \frac{\vec{P}_0 \cdot (\vec{k} \times \vec{N})}{|\vec{k} \times \vec{N}|}, \text{ if } |\vec{P}_0| = 1.$$

NOTE: All development was based on the solution to second order differential equations.

The sphere was treated with both condition types, ie,

Case 1:

DE: Sphere

BC: $\vec{P}(t = 0) = \vec{P}_0$, $\vec{P}(t = \tau) = \vec{P}_1$ (boundary value problem).

Case 2:

DE: Sphere

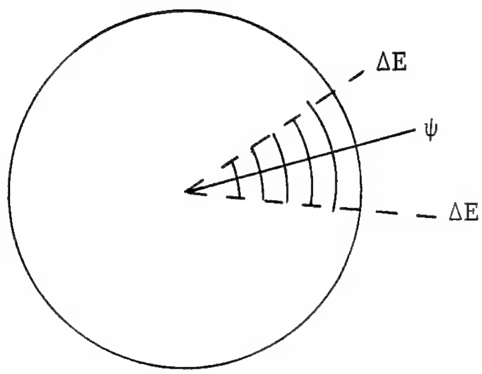
IC: $\vec{P}(t = 0) = \vec{P}_0$, $\frac{d\vec{P}}{dt}(t = 0) = \tan\psi$ (initial condition problem).

In both cases, it was convenient to find $\vec{N}$, the planar normal.

Both cases have physical interest since case 1 is the classical navigation problem, case 2 can be thought as: given the position and direction of an enemy aircraft, where is it going, what will it most likely attack, minima curve assumed.

In the second case, an uncertainty in ψ (ie, surface directional heading, where $\psi = \psi \pm \Delta E$) will generate a solution envelope.

The solution envelope is easily generated if $\vec{P}_0$ is rotated to the north pole.



The solution obtained is a surface envelope (or sectoral slice) which can be useful in determining probable targets.

(4) Selected problems.

(a) Intersection of minima distance curves. In the plane, the following expressions can be written for the two minima trajectories:

$$y = ax + b$$

$$y' = cx' + d$$

The objective is to find the minima trajectories when

$$y = y',$$

$$x = x'$$

hence,

$$ax + b = cx + d$$

$$(a - c)x = d - b$$

therefore,

$$x = x' = \frac{d - b}{a - c} \qquad (a \neq c)$$

$$y = y' = a\left(\frac{d - b}{a - c}\right) + b = c\left(\frac{d - b}{a - c}\right) + d$$

$$y' = \left(\frac{1}{a - c}\right)(ad - ab + ab - cd) = \frac{1}{a - c}(ad - cb)$$

$$y = \left(\frac{1}{a - c}\right)(cd - cb + da - cd) = \frac{1}{a - c}(ad - cb) \quad \text{(results are consistent).}$$

NOTE: If $a = c$ there is: no solution if $b \neq d$; no unique solution if $b = d$.

Now, no unique solution implies $a = c$ and $b = d$, ie, the trajectories are on the same curve.

No solution implies y and y' are nonintersecting parallel lines.

On the sphere there exists similar cases.

Let

$\vec{N} \cdot \vec{x} = 0$ (one curve).

$\vec{N}' \cdot \vec{x} = 0$ (another curve).

The vector which points to the intersection of the curves is easily seen to be:

NOTE: The intersection vector $\vec{I}$ must reside in the plane $\vec{N}$, hence:
$\vec{I} \cdot \vec{N} = 0$, but $\vec{I}$ must also reside in $\vec{N}'$ so
$\vec{I} \cdot \vec{N}' = 0$, so that $\vec{I}$ is orthogonal to both $\vec{N}$, $\vec{N}'$ therefore
$\vec{I} \propto \vec{N} \times \vec{N}'$, now since $|\vec{I}| = a$ (if $\vec{I} \neq 0$), $\vec{I}$ can be set equal to $\vec{N} \times \vec{N}'$, then let, $|\vec{I}| = 1$. $\vec{I}$ now points to the latitude/longitude of the intersection/intersections (remember there are two, $\vec{I}$ and $-\vec{I}$).

$\vec{I} = \vec{N} \times \vec{N}'$ (since $\vec{I}$ must be a vector in both planes).

$\vec{I}$ is unique if and only if

$\vec{N} \times \vec{N}' \neq 0$.

If $\vec{N} \times \vec{N} = 0$, then either $\vec{N} = \vec{N}'$ or $\vec{N} = -\vec{N}'$ which is similar to case 2 in the plane, no unique solution.*

There is no analog to planar case 1 (no solution) on the sphere.

(b) Trajectory/intercept. In Cartesian R^3 let the invader parameters be,

$\vec{P}_0(t) = \vec{P}_0 + \vec{V}_0 t$,

and the interceptor parameters be

*The trajectories are on the same planar/sphere intersection curve.

$\vec{P}_1(t) = \vec{P}_1 + \vec{V}_1 t$

where $\vec{V}_1$, $\vec{V}0$ are assumed constants.

Then at some time ($t = \tau$) it is required that

$\vec{P}_0(\tau) = \vec{P}_1(\tau)$.

The object is to solve for the components of $\vec{V}_1$, given $\vec{P}_0$, $\vec{P}_1$ and $\vec{V}_0$.

Let,

$\vec{P}_0 = x_0, y_0, z_0$

$\vec{P}_1 = x_1, y_1, z_1$

$\vec{V}_0 = V_{x0}, V_{y0}, V_{z0}$

$\vec{V}_1 = V_{x1}, V_{y1}, V_{z1}$

$\left(|V_{x1}^2 + V_{y1}^2 + V_{z1}^2| = V_1^2\right.$, and it is assumed $|\vec{V}_1|$ known$\left.\right)$.

Now at $t = \tau$

$x_0 + V_{x0}\tau = x_1 + v_{x1}\tau$

$y_0 + V_{y0}\tau = y_1 + V_{y1}\tau$

$z_0 + V_{z0}\tau = z_1 + V_{z1}\tau$

isolating the V_1 terms:

$V_{x1} = \frac{1}{\tau}\left(x_0 - x_1 + V_{x0}\tau \right)$

$V_{y1} = \frac{1}{\tau}\left(y_0 - y_1 + V_{y0}\tau \right)$

$V_{z1} = \frac{1}{\tau}\left(z_0 - z_1 + V_{z1}\tau \right)$ (16)

hence we have a quadratic equation in τ

$$V_1^2\tau^2 = (x_0 - x_1 + V_{x0}\tau)^2 + (y_0 - y_1 + V_{y0}\tau)^2 + (z_0 - z_1 + V_{z0}\tau)^2$$

$$V_1^2\tau^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 + V_0^2\tau^2 + 2(x_0 - x_1)V_{x0}\tau$$
$$+ 2(y_0 - y_1)V_{y0}\tau + 2(z_0 - z_1)V_{z0}\tau$$

$$(V_1^2 - V_0^2)\tau^2 = |\vec{P}_0 - \vec{P}_1|^2 + 2(\vec{P}_0 - \vec{P}_1)\cdot\vec{V}_0\tau$$

or, assuming $\left(V_1^2 > V_0^2\right)$ this can be rewritten as:

$$a\tau^2 - 2b\tau - c = 0$$

$$a = \left(V_1^2 - V_0^2\right)$$

$$b = (\vec{P}_0 \quad \vec{P}_1)\cdot\vec{V}_0$$

$$c = |\vec{P}_0 - \vec{P}_1|^2$$

which has solutions:

$$\tau = \frac{1}{a}\left[b \pm (b^2 + ac)^{1/2}\right] \quad .$$

Assume $\tau > 0$, so the solution is physically real, therefore

$$\tau = \left(V_1^2 - V_0^2\right)^{-1}\Big[(\vec{P}_0 - \vec{P}_1)\cdot\vec{V}_0 + \Big[|(\vec{P}_0 - \vec{P}_1)\cdot\vec{V}_0|^2$$
$$+ |\vec{P}_0 - \vec{P}_1|^2\left(V_1^2 - V_0^2\right)\Big]^{1/2}\Big]$$

The components for V_1 are found via substituting τ into equation (16).

Thus, the values of both $\vec{V}_1$ and τ, the time of intercept are obtained.

Now if $\left(V_1^2 \leqslant V_0^2\right)$, a physically real solution can only exist

$$\text{if}(\vec{P}_0 - \vec{P}_1)\cdot\vec{V}_0 < 0 \quad .$$

For example, let

$$V_1^2 = V_0^2$$

then ,

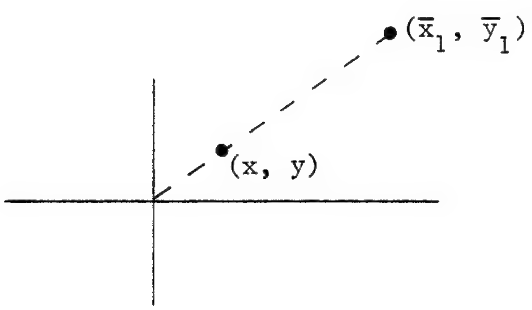
$$\tau = \frac{|\vec{P}_0 - \vec{P}_1|^2}{-2\left(\vec{P}_0 - \vec{P}_1\right) \cdot \vec{V}_0}$$ (since $|\vec{P}_0 - \vec{P}_1|^2 \geq 0$, then for a solution to exist

$\left(\vec{P}_0 - \vec{P}_1\right) \cdot \vec{V}_0 < 0$.

(The case of $V_0^2 > V_1^2$, the discriminant must be real for a solution to exist.)

(c) Curve of pursuit.

NOTE: Although this section works the curve of pursuit problem, the interception problem on a sphere is also addressed here. This is because the solution technique is the same (or very similar) for both problems.



Now,

$$\frac{dy}{dx} = \frac{\bar{y}_1 - y}{\bar{x}_1 - x}$$

ie, let

$y_1 = y_{10} + V_y t$

$x_1 = x_{10} + V_x t$

and let

$V_{ox} = V_0 \cos \theta$

$V_{oy} = V_0 \sin \theta$

then set a parametric relation for x(t), y(t).

$$\dot{y}(t) = \frac{\bar{y} - y}{r} V$$

$$\dot{x}(t) = \frac{\bar{x} - x}{r} V$$

where:

$$r^2 = (\bar{x} - x)^2 + (\bar{y} - y)^2$$

$V = |V|$, the velocity (assumed constant) of the interceptor.

The idea is the instantaneous slope (or direction) of the interceptor points along the line (curve of minimum distance) which includes both interceptor and target instantaneous positions. As with most differential equations of this type exact solutions depend on $[\bar{x}(t), \bar{y}(t)]$ the target instantaneous position and the initial conditions.

On the sphere, $\bar{\vec{N}}$ refers to the target and $\vec{N}$ to the interceptor. As $\bar{\vec{P}}$ of the target changes, iterate on $\vec{P}$ and $\vec{N}$ of the interceptor. It is convenient (for $\bar{\vec{N}}$ = constant target on a minima course) to use the matrix concepts developed earlier. Treat this case analgous to target

$$\bar{y} = \bar{y}_0 + V_y t$$

$$\bar{x} = \bar{x}_0 + Vt_x \quad .$$

The instantaneous $\vec{N}$ is always $\bar{\vec{P}} \times \vec{P}$. This is used as the motion in Δt is divided up.

Say for example the given terms are: $\bar{\vec{P}}_0$, $\vec{P}$, $\left(|\bar{V}|, |V|\right)$, $\bar{\vec{N}}$, $\vec{N}_0 = \vec{P}_0 \times \bar{\vec{P}}_0$, then calculate:

$$\bar{\vec{P}}_1 = \bar{M} \begin{pmatrix} \cos\left(\delta + \frac{|\bar{V}|}{a} \Delta t\right) \\ \sin\left(\delta + \frac{|\bar{V}|}{a} \Delta t\right) \\ 0 \end{pmatrix}$$ [remember $\bar{M} = \bar{M}(\bar{\vec{N}})$; M, related to the α, β matrix, reference IVb(3)(a)2].

and

$$\vec{P}_1 = M \begin{pmatrix} \cos\left(\delta \pm \frac{|V|}{a}\right)\Delta t \\ \sin\left(\delta \pm \frac{|V|}{a}\right)\Delta t \\ 0 \end{pmatrix}$$

$M = M(\vec{N})$

then recalculate $\vec{N}$,

$$\vec{N}_1 = \vec{P}_1 \times \vec{\bar{P}}_1$$

Since $\vec{\bar{N}}$ is constant, $\bar{M}$ is constant and

$$\vec{\bar{P}}_n = \bar{M} \begin{pmatrix} \cos\left(\delta + n \frac{|\bar{V}|}{a} \Delta t\right) \\ \sin\left(\delta + n \frac{|\bar{V}|}{a} \Delta t\right) \\ 0 \end{pmatrix}$$

Now, since $\vec{N}$ changes for each Δt cycle, $M(\vec{N})$ changes and for a given cycle (call it M)

$$\vec{N}_m = \vec{P}_m \times \vec{\bar{P}}_m$$

$$M_m = M(\vec{N}_m)$$

$$\vec{\rho}_{m+1} = M_m \begin{pmatrix} \cos\left(\delta_m \pm \frac{|V|}{a} \Delta t\right) \\ \sin\left(\delta_m \pm \frac{|V|}{a} \Delta t\right) \\ 0 \end{pmatrix}$$

and continue the cycle.

As in the planar case, the spherical case is easily generalized to $\vec{\bar{P}}_0 = \vec{\bar{P}}_{(n\Delta t)}$

for $\vec{\bar{P}}$ a general curve.

NOTE: The $\vec{P}$ terms are $\vec{P}\left(\delta \pm \frac{|V|}{a} \Delta t\right)$. The $\pm$ can be resolved by requiring $\vec{P}_1 \cdot \bar{\vec{P}}_1 > \vec{P}_0 \cdot \bar{\vec{P}}_0$ or $\vec{P}_{m+1} \cdot \bar{\vec{P}}_{m+1} > \vec{P}_m \cdot \bar{\vec{P}}_m$, so the interceptor always points towards the target.

(Assuming $|V| > |\bar{V}|$).

(d) The spherical interceptor problem.

Given $\vec{P}_0$, $\vec{N}_0$ (and V_0) and $\vec{P}_1$ (and V_1) find $\vec{N}_1$, the intercept path.

Let $\vec{P}_I$ be the point of interception.

For the two vectors $\vec{A}$ and $\vec{B}$ and a unitary matrix M

$\vec{A} \cdot \vec{B} = M\vec{A} \cdot M\vec{B}$ (since only the relative orientation is changed).

Now let

$$\vec{P}_I = M \begin{pmatrix} \cos \psi_I \\ \sin \psi_I \\ 0 \end{pmatrix}$$

$$\vec{P}_0 = M \begin{pmatrix} \cos \delta \\ \sin \delta \\ 0 \end{pmatrix}$$

[M, the α, β matrix defined so as to make $\left(M^{-1}\vec{P}_0\right) \cdot \hat{k} \equiv 0$.]

($\vec{N}$ not proportional to $\vec{P}_0 \times \vec{P}_1$ has been implicity assumed. The assumption should be apriori checked by $\dfrac{\vec{P}_0 \times \vec{P}_1}{|\vec{P}_0 \times \vec{P}_1|} \neq \dfrac{\vec{N}}{|\vec{N}|}$.)

Hence

$$M^{-1}\vec{P}_I = \begin{pmatrix} \cos\psi_I \\ \sin\psi_I \\ 0 \end{pmatrix}$$

$$M^{-1}\vec{P}_0 = \begin{pmatrix} \cos\delta \\ \sin\delta \\ 0 \end{pmatrix} \qquad (\delta \text{ known}).$$

Continue with

$$M^{-1}\vec{P}_0(\tau) = \begin{pmatrix} \cos(\delta + \omega\tau) \\ \sin(\delta + \omega\tau) \\ 0 \end{pmatrix} \qquad \left(\omega = \frac{V_0}{a}\right) .$$

$$M^{-1}\vec{P}_1 = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \qquad (\text{fixed})^*.$$

Then find τ such that:

$$M^{-1}P_0(\tau) \cdot \bar{M}^{1}P_0(0) = \cos\frac{V_0}{a}\tau \qquad (\text{target equation})$$

and

$$M^{-1}\vec{P}_1 \cdot \bar{M}^{1}P_0(\tau) = \cos\frac{V_1}{a}\tau \qquad (\text{interceptor equation}).$$

$M^{-1}\vec{P}_0(0)$ is defined as $\begin{pmatrix} \cos\delta \\ \sin\delta \\ 0 \end{pmatrix}$, so the first equation is always true.

The second equation becomes

*$\vec{P}_0$ the target vector $\vec{P}_1$, the interceptor vector.

$$a' \cos\left(\delta + \frac{V_0}{a}\tau\right) + b' \sin\left(\delta + \frac{V_0}{a}\tau\right) = \cos\left(\frac{V_1}{a}\tau\right).$$

Now since (for physical solutions) $\tau \geqslant 0$, the $x = f(x)$ type iteration solution can be used:

$$\tau = \frac{a}{V_1}\cos^{-1}\left[a' \cos\left(\delta + \frac{V_0}{a}\tau\right) + b' \sin\left(\delta + \frac{V_0}{a}\tau\right)\right] ,$$

starting with $\tau = 0$ and continuing until some convergence criteria is met.

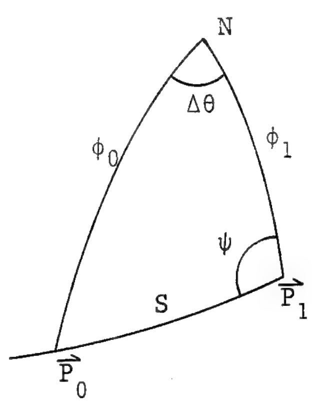
Then with τ solved,

$$\vec{P}_0(\tau) = \vec{P}_I \quad .$$

With the point of interception found $\left(\vec{P}_I \text{ known}\right)$ the interceptor can be navigated to the target on the $\vec{N} = \vec{P}_1 \times \vec{P}_I$ plane.

(5) Directional range and location problem.

Given $\vec{P}_0$, s, ψ find $\vec{P}_1$: (eg, let P_0 be a radio beacon at a known location. Given a radio receiver at $\vec{P}_1$ which can determine it's distance, s, from $\vec{P}_0$ and the direction (ψ) to $\vec{P}_0$, where is the receiver located, determine $\vec{P}_1$.)



$\vec{P}_0$ has coordinates ϕ_0, θ_0, and $\vec{P}_1$ has coordinates ϕ_1, θ_1; ϕ_0, θ_0, ψ and s are known.

Now

$$\frac{\sin \psi}{\sin \phi_0} = \frac{\sin \Delta\theta}{\sin s}$$

so that

$$\theta_1 = \theta_0 + \sin^{-1}\left(\sin s \frac{\sin \psi}{\sin \phi_0}\right) \qquad \left(\theta_1 \text{ determined}\right).$$

NOTE: Quadrant relationship ψ to $\Delta\theta$ positive.

Also:

$$\cos s = \cos \phi_0 \cos \phi_1 + \sin \phi_0 \sin \phi_1 \cos \Delta\theta$$

$$\cos \phi_0 = \cos \phi_1 \cos s + \sin \phi_1 \sin s \cos \psi$$

$$\sin \phi_1 = \frac{\cos s - \cos \phi_0 \cos \phi_1}{\sin \phi_0 \cos \Delta\theta}$$

$$\cos \phi_0 = \cos s \cos \phi_1 + \left(\frac{\sin s \cos \psi}{\sin \phi_0 \cos \Delta\theta}\right) \left(\cos s - \cos \phi_0 \cos \phi_1\right)$$

$$\cos \phi_0 - \cos s \left(\frac{\sin s \cos \psi}{\sin \phi_0 \cos \Delta\theta}\right) = \cos \phi_1 \left(\cos s - \frac{\sin s \cos \psi \cos \phi_0}{\sin \phi_0 \cos \Delta\theta}\right)$$

$$\cos \phi_1 = \frac{\cos \phi_0 \sin \phi_0 \cos \Delta\theta - \cos s \sin s \cos \psi}{\cos s \sin \phi_0 \cos \Delta\theta - \cos \phi_0 \sin s \cos \psi} \qquad \left(\phi_1 \text{ determined}\right).$$

$$\cos \phi_1 = \frac{\sin 2\phi_0 \cos \Delta\theta - \sin 2s \cos \psi}{2 \cos s \sin \phi_0 \cos \Delta\theta - \cot \phi_0 \tan s \cos \psi}$$

NOTE: On an oblate planet, if it can be assumed the locator beacon travels the minimum distance route, finding $\vec{P}_1$, given $\vec{P}_0$, ψ, s is an iterative process. Start with the sphere relations and iterate using oblate geodesic curve relations. Solution is unique as long as $\vec{P}_0$ is not a pole (north or south).

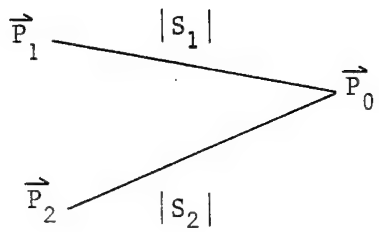
$$\cos s \sin \phi_0 \cos \Delta\theta \neq \cos \phi_0 \sin s \cos \psi$$

but if $\phi_0 = 0$ then $\cos \phi_1 = \cos s$, $\Delta\theta$ undeterminable.

(6) Given range, calculate location problem.

(a) On a plane.

Given $\vec{P}_1$, $\vec{P}_2$, $|s_1|$, $|s_2|$ locate P_0.



Let

$$\vec{P}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\vec{P}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

and assume $\vec{P}_0$, $\vec{P}_i$ unique, so that

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = s_1^2$$

$$(x_2 - x_0)^2 + (y_2 - y_0)^2 = s_2^2$$

. Two equations in two unknowns (note, there may be two solutions, one solution or no solutions).

Now

$$x_0 = x_2 \pm \left[s_2^2 - (y_2 - y_0)^2\right]^{1/2}$$

so that

$$x_1 - x_2 = \pm \left[s_2^2 - (y_2 - y_0)^2\right]^{1/2} \pm \left[s_1^2 - (y_1 - y_0)^2\right]^{1/2}$$

and solve for y_0 (then for x_0).

(b) On a sphere.

Let

$$\vec{P}_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\vec{P}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix},$$

$$\vec{P}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

$\vec{P}_0 \cdot \vec{P}_1$ = known = d_1

$\vec{P}_0 \cdot \vec{P}_2$ = known = d_2

$|\vec{P}_0|$ = known = 1

hence, assuming $|\vec{P}_i| = 1$

$\vec{P}_1$, $\vec{P}_2$ given, solve for $\vec{P}_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

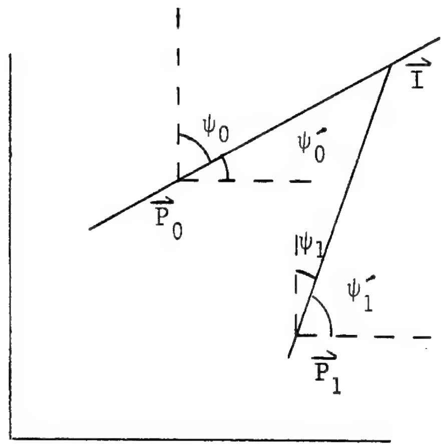
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ x & y & z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ 1 \end{pmatrix}.$$

(x, y, z easily solved for).

(7) Triangulation problem. Given $\vec{P}_0$, $\vec{P}_1$, ψ_0 and ψ_1 find $\vec{I}$, the point of interception.

(a) On a plane.

Trivial:



$$\psi_i' = \tan^{-1} \frac{dy}{dx}$$

Let

$$\frac{1}{a_i} = \tan \psi_i$$

$$b_i = y_i - a_i x_i$$

where

$$\vec{P}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

hence

$$y = a_0 x + b_0 \quad (\overrightarrow{P_0 I})$$

$$\bar{y} = a_1 \bar{x} + b_1 \quad (\overrightarrow{P_1 I}).$$

Let

$$I = \begin{pmatrix} \bar{y} = y \\ \bar{x} = x \end{pmatrix}$$

$$a_0 x + b_0 = a_1 x + b_1$$

and

$$\bar{\bar{x}} = \frac{b_1 - b_0}{a_0 - a_1} \quad \left(\text{provided } a_0 \neq a_1 \right)$$

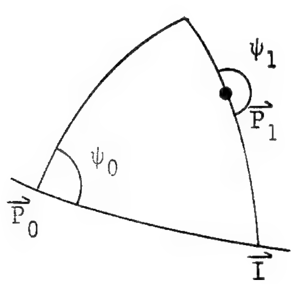
$$\bar{\bar{y}} = a_0 x + b_0 = a_1 \bar{x} + b_1 \quad \text{(either relation).}$$

Hence

$$\vec{I} \text{ solved for: } \begin{pmatrix} \bar{\bar{x}} \\ \bar{\bar{y}} \end{pmatrix} .$$

(b) On a sphere.

Also trivial, look:



As previously done [reference IVb(2)(b)] $\vec{P}_i$, ψ_i generates $\vec{N}_i$ and $\vec{I}$ is generated as:

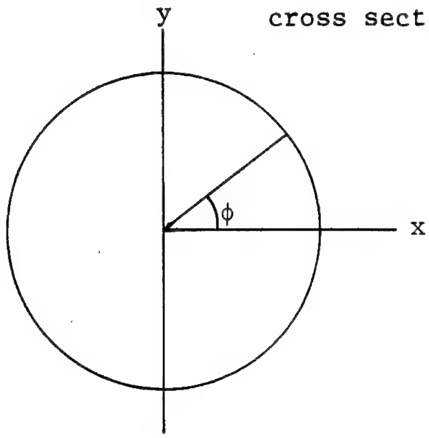
$$\vec{I} = \vec{N}_0 \times \vec{N}_1 \quad ,$$

then setting $|\vec{I}| = 1$ obtain $\{\phi, \theta\}$ directly. Always make sure the quadrant is correct as $\vec{N}_0 \times \vec{N}_1$ points to $\pm \vec{I}$.

c. Application to the slightly oblate spheroid. First order perturbation techniques are used throughout.

(1) Surface modeling. The zero order approximation is that of a sphere.

$$x^2 + y^2 + z^2 = a^2 \text{ or } \rho = a \quad .$$



Working with the cross section: $x^2 + y^2 = a^2$.

For the first order planetary shape approximation, the assumption is made that the surface normal points roughly in the direction of the acceleration vector.

(a) Nonrotating surface.

$\vec{N} = \nabla \left(x^2 + y^2 - a^2 \right)$,

let

$|\vec{N}| = g_r$

$\vec{N} = \frac{g_r}{R} \left(x\hat{i} + y\hat{j} \right)$, R a scaler constant.

Also,

$$\tan \phi = - \frac{dx}{dy} = \frac{\frac{g_r}{R}\, y}{\frac{g_r}{R}\, x}$$

or, the y acceleration component divided by the x component.

(b) For a rotating surface. The x component is reduced by an amount $-\omega^2 x$ due to centrifugal acceleration. If the resulting surface is still to remain approximately normal to the acceleration vector, then

$$-\frac{dx}{dy} \cong \frac{\frac{g_r}{R}\, y}{\left(\frac{g_r}{R} - \omega^2\right) x}$$

yielding

$$\left(1 - e^2\right) x^2 + y^2 = b^2 \quad \left(e^2 = \frac{\omega^2 R}{g_r}\right)$$

which is of the form of an oblate spheroid.

Now, for a solid body

$0 \leqslant g \leqslant g_R,$

since $g = g(r)$ (assume symmetric, isotropic body).

$g(r) \cong \frac{4\pi G}{3} \bar{\rho} r$ $\quad \bar{\rho}$ = average density (assume symmetric, isotropic body).

As a first guess (and trying to strike a compromise between a solid body and a thin surface) let $g(r) = \frac{1}{2} g_R$* the arithmetic average of $g(r)$ on r, from $0 \leqslant r \leqslant R$, R the approximate radius of the spheroid volume.

Then substituting back:

$$< e^2 > \cong \frac{3\omega^2}{2\pi G\bar{\rho}} \quad .$$

For the earth

$\omega^2 \cong 5.3176 \cdot 10^{-9}$ (radians/second)^2

$G = 6.673 \cdot 10^{-11}$ N M^2/kg^2

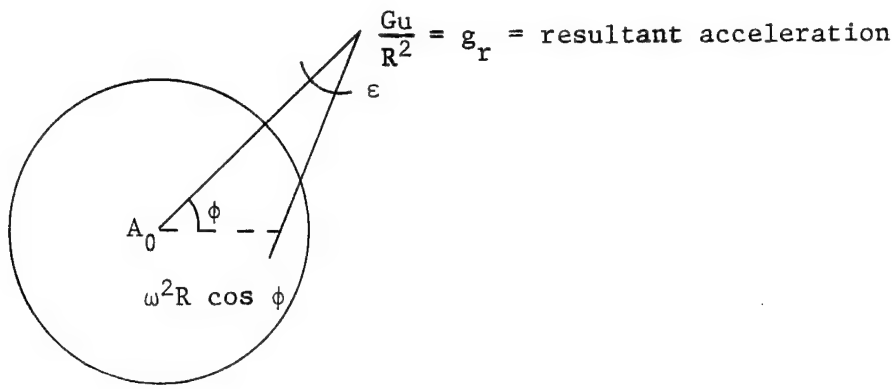
$\bar{\rho} \cong 5.522$ gcm^{-3}

yielding

$< e^2 > \cong 0.00689$.

The plumb line is deflected as follows:

*The same value is obtained via $< g_R > = \frac{1}{R} \int_0^R g(r)\, dr$.



and

$$\frac{\sin \varepsilon}{\omega^2 R \cos \phi} = \frac{\sin \phi}{g_R}$$

hence[6]

$$\varepsilon \sim \frac{\omega^2}{2} \sin 2\phi \, \frac{R}{g_R} \sim \frac{\omega^2 R \sin 2\phi}{2 g_R} \quad .$$

For an alternative derivation of $\langle e^2 \rangle$, dimensional analysis[7] can be used.

From the Buckingham pi theorum choose the following pi:

π_0 = e, eccentricity (a pure number).

π_1 = ω, angular/rotation velocity.

π_2 = ρ, density.

π_3 = G, gravitational constant (although a constant, watch!!).

Start as follows:

$$F = \frac{G\mu m}{R^2}$$

[6] Fowles, Analytical Mechanics, P. 130.

[7] Hughes, Schaum's Outline, Fluid Mechanics, Chapter 5.

$$a = \omega^2 R = \frac{G\mu}{R^2}$$

$$\omega^2 = G\left(\frac{\mu}{R^3}\right) = G\rho \quad .$$

Assume ω in the numerator (this seems feasable as ω↑, e↑) then going back to the pi theorum:

$$e = k\left(\frac{\omega^2}{G\rho}\right)^n \quad .$$

NOTE: This also assumes $n > 0$.

If $n \equiv \frac{1}{2}$ is chosen, then the value for k is

$$k_{\text{theoretical}} = \left(\frac{3}{2\pi}\right)^{1/2} \cong 0.69099 \quad \left(< e_t > = 0.08301 \right).$$

$$k_{\text{actual}} \cong 0.68253 \quad \left(< e_a > \cong 0.081992 \right) \quad .$$

Thus giving a percent differential for k, $\left(100 \cdot \frac{k_t - k_a}{k_t}\right) \cong 1.22\%!!$

(2) Oblate spheroid. According to 1975 CRC[8] for the earth.

$a = 6378.388$ km

$b = 6356.912$ km

yielding

$$e^2 \cong 0.00672 \quad .$$

NOTE: These sections are not meant to serve as a rigorous derivation for e, the planetary eccentricity value, but more as a heuristic demonstration on how to arrive at an approximate value.

Consider the surface model to be a slightly oblate spheroid with eccentricity $e < 1$, $e^2 \ll 1$. In the analysis consider only terms up to and including e^2. All other terms shall be discarded as being vanishingly small. Continue with:

$$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1 \text{ (letting } r = \rho \sin\phi \text{ and } z = \rho\cos\phi).$$

$$\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2} = \frac{1}{\rho^2}$$

[8]Consolidated Rubber Company (CRC Handbook of Mathematical Tables, 1975).

$$\rho^2 = a^2 b^2 \left(b^2 \sin^2 \phi + a^2 \cos^2 \phi \right)^{-1}$$

$$\rho^2 = b^2 \left(\frac{b^2}{a^2} \sin^2 \phi + \cos^2 \phi \right)^{-1}$$

$$\rho^2 = b^2 \left(1 - \left(1 - \frac{b^2}{a^2} \right) \sin^2 \phi \right)^{-1} \qquad \left(\text{let } 1 - \frac{b^2}{a^2} = e^2 \right) .$$

$$\rho = b \left(1 - e^2 \sin^2 \phi \right)^{-1/2} .$$

Now, assume $e^2 << 1$.

$$\rho \cong b \left(1 + \frac{e^2}{2} \sin^2 \phi \right)$$

and

$$\left| \frac{d\rho}{d\phi} \right|^2 \sim \text{order of } e^4$$

set

$$\left| \frac{d\rho}{d\lambda} \right|^2 \sim 0 \left(\text{since } \frac{d\rho}{d\lambda} = \frac{d\rho}{d\phi} \frac{d\phi}{d\lambda} \right) .$$

Continue with

$$s \cong \int b \left(1 + \frac{e^2}{2} \sin^2 \phi \right) \left[\left(\frac{d\phi}{d\lambda} \right)^2 + \sin^2 \phi \left(\frac{d\theta}{d\lambda} \right)^2 \right]^{1/2} d\lambda .$$

As in the case of the sphere let

$$\phi = \tan^{-1} \lambda$$

$$\frac{d\phi}{d\lambda} = \frac{1}{1 + \lambda^2}$$

$$\sin^2 \phi = \frac{\lambda^2}{1 + \lambda^2} ,$$

so

$$s = b \int \left(1 + \frac{e^2 \lambda^2}{2 \left(1 + \lambda^2 \right)} \right) \left(\frac{1}{\left(1 + \lambda^2 \right)^2} + \frac{\lambda^2}{1 + \lambda^2} g'^2 \right)^{1/2} d\lambda \quad \left(g' = \frac{d\theta}{d\lambda} \right)$$

since

$$\frac{\partial \delta}{\partial g} = 0 \cdot ,$$

let

$$\frac{\partial \delta}{\partial g'} = c$$

so that:

$$\left[1 + \frac{e^2\lambda^2}{2\left(1+\lambda^2\right)}\right] \frac{\dfrac{\lambda^2}{1+\lambda^2}\, g'}{\left[\dfrac{1}{\left(1+\lambda^2\right)^2} + \dfrac{\lambda^2}{1+\lambda^2}\, g'^2\right]^{1/2}} = c$$

letting

$$c' = c\left(1 + \frac{e^2\lambda^2}{2\left(1+\lambda^2\right)}\right)^{-1} \quad ,$$

$$c'^2 \cong c^2\left(1 - \frac{e^2\lambda^2}{\left(1+\lambda^2\right)}\right)$$

$$\left(\frac{\lambda^2}{1+\lambda^2}\, g'\right)^2 = c'^2\left(\frac{1}{\left(1+\lambda^2\right)^2} + \frac{\lambda^2}{1+\lambda^2}\, g'^2\right)$$

$$\lambda^4 g'^2 = c'^2 + c'^2\lambda^2\left(1+\lambda^2\right) g'^2$$

$$\lambda^4 g'^2 \doteq c^2 - c^2e^2\,\frac{\lambda^2}{1+\lambda^2} + c^2\lambda^2\left(1+\lambda^2\right)g'^2 - c^2e^2\lambda^4 g'^2$$

$$(1 + c^2e^2)\lambda^4 g'^2 = c^2\left(1 - \frac{e^2\lambda^2}{1+\lambda^2}\right) + c^2\lambda^2(1+\lambda^2)g'^2$$

$$\lambda^2\left[\lambda^2(1 + c^2e^2) - c^2(1+\lambda^2)\right]g'^2 = c^2\left(1 - \frac{e^2\lambda^2}{1+\lambda^2}\right)$$

$$\lambda^2\left[\lambda^2 + \lambda^2c^2e^2 - c^2 - c^2\lambda^2\right]g'^2 = c^2\left(1 - \frac{e^2\lambda^2}{1+\lambda^2}\right)$$

$$\lambda^2\left[\lambda^2(1 + c^2e^2 - c^2) - c^2\right]g'^2 = c^2\left(1 - \frac{e^2\lambda^2}{1+\lambda^2}\right)$$

$$\lambda^2 \left[\frac{\lambda^2}{\beta^2} - 1\right] g'^2 = \left(1 - \frac{e^2\lambda^2}{1 + \lambda^2}\right)$$

$$\frac{1}{\beta^2} = \frac{1}{c^2}\left(1 - c^2 + e^2c^2\right)$$

$$\lambda \left[\frac{\lambda^2}{\beta^2} - 1\right]^{1/2} g' \cong (\pm)\left(1 - \frac{e^2\lambda^2}{2\left(1 + \lambda^2\right)}\right)$$ (again expanding on e^2).

NOTE: The second (e^2) term acts like a perturbing term which always detracts from the great circle route.

Now, assuming the positive solution:

$$g' = \frac{1}{\lambda\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2}} - \frac{e^2}{2}\frac{\lambda}{\left(1 + \lambda^2\right)\left(\frac{\lambda^2}{\beta} - 1\right)^{1/2}}$$

$$g(\lambda) = \sec^{-1}\frac{\lambda}{\beta} - \frac{e^2}{2} q(\lambda)$$ ($q(\lambda)$ a perturbing term).

$$q(\lambda) = \int \frac{\lambda d\lambda}{\left(1 + \lambda^2\right)\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2}} .$$

Let

$$\mu = 1 + \lambda^2$$

$$du = 2\lambda\, d\lambda$$

$$\lambda^2 = \mu - 1$$

$$\frac{\lambda^2}{\beta^2} = \frac{1}{\beta^2}(\mu - 1)$$

$$q(\lambda) = \frac{\beta}{2}\int \frac{du}{\mu(\mu - \alpha)^{1/2}} \qquad \left(\alpha = 1 + \beta^2\right)$$

$$q(\lambda) = \frac{\beta}{2}\left[\frac{2}{(\alpha)^{1/2}} \tan^{-1}\left(\frac{\mu - \alpha}{\alpha}\right)^{1/2}\right]$$

$$q(\lambda) = \frac{\beta}{\left(1 + \beta^2\right)^{1/2}} \tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2}$$

$$g(\lambda) = \theta_0 \pm \left\{ \sec^{-1}\left(\frac{\lambda}{\beta}\right) - \frac{e^2}{2} \frac{\beta}{(1+\beta^2)^{1/2}} \tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} \right\} \quad (\lambda \geqslant \beta).$$

NOTE: $$g' = \frac{1}{\lambda\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2}} \left(1 - \frac{e^2}{2} \frac{\lambda^2}{1+\lambda^2}\right).$$

So for the first three equations:

$$\rho = b\left(1 + \frac{e^2}{2} \frac{\lambda^2}{1+\lambda^2}\right)$$

$$\phi = \tan 1^{-1} \lambda$$

$$\theta = \theta_0 \pm \left\{ \sec^{-1} \frac{\lambda}{\beta} - \frac{e^2}{2} \frac{\beta}{(1+\beta^2)^{1/2}} \tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} \right\} \quad (\lambda \geqslant \beta).$$

For the sphere the nodal crossings:

$$\theta_{(\lim \lambda \to \infty)} = \theta_0 \pm \frac{\pi}{2}.$$

For the slightly perturbed spheroid

$$\theta_{(\lim \lambda \to \infty)} = \theta_0 \pm \frac{\pi}{2}\left(1 - \frac{e^2}{2} \frac{\beta}{(1+\beta^2)^{1/2}}\right).$$

The new curve is slightly crunched up, presumably to avoid the equatorial budge.

The distance equation becomes:

$$s \cong b\int\left(\left(1 + \frac{e^2}{2} \cdot \frac{1}{1+\lambda^2}\right)\left[\frac{1}{(1+\lambda^2)^2} + \frac{\lambda^2}{1+\lambda^2}\left(\frac{\beta^2}{\lambda^2(\lambda^2-\beta^2)}\right.\right.\right.$$

$$\left.\left.\left.\left(1 - e^2 \frac{\lambda^2}{1+\lambda^2}\right)\right)\right]^{1/2} d\lambda$$

$$s = b\int \frac{1}{1+\lambda^2}\left(1 + \frac{e^2}{2} \frac{\lambda^2}{1+\lambda^2}\right)\left[1 + \frac{(1+\lambda^2)\beta^2}{\lambda^2-\beta^2}\left(1 - e^2 \frac{\lambda^2}{1+\lambda^2}\right)\right]^{1/2} d\lambda$$

$$s = b\int \frac{1}{1+\lambda^2} \cdot \frac{1}{(\lambda^2-\beta^2)^{1/2}}\left(1 + \frac{e^2}{2} \frac{\lambda^2}{1+\lambda^2}\right)$$

$$\cdot \left[\lambda^2 - \beta^2 + \beta^2 + \beta^2\lambda^2 - e^2\lambda^2\beta^2\right]^{1/2} d\lambda$$

$$s = b\int \frac{\lambda}{1+\lambda^2} \cdot \frac{d\lambda}{(\lambda^2 - \beta^2)^{1/2}} \left(1 + \frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right)\left(1 + \beta^2 - e^2\beta^2\right)^{1/2}$$

$$s = b\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2} \int \left(1 + \frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right) \frac{\lambda d\lambda}{\left(1+\lambda^2\right)\left(\lambda^2 - \beta^2\right)^{1/2}}$$

let,

$$b' = b\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2}$$

$$s = b' \int \frac{\lambda d\lambda}{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}} + b'\frac{e^2}{2}\int \frac{\lambda^2}{1+\lambda^2} \cdot \frac{\lambda d\lambda}{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}} \ .$$

The first integral solves as

$$= b' \frac{1}{\left(1+\beta^2\right)^{1/2}} \tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} \ .$$

For the second integral

$$= b'\frac{e^2}{2}\int \frac{\lambda^2}{1+\lambda^2} \cdot \frac{\lambda d\lambda}{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}} \ .$$

Let

$$\mu = 1 + \lambda^2$$

$$du = 2\lambda d\lambda$$

$$= b'\frac{e^2}{4}\int \frac{\mu - 1}{\mu} \cdot \frac{du}{\mu\left(\mu - \alpha\right)^{1/2}}$$

$$= b'\frac{e^2}{4}\int \frac{du}{\mu\left(\mu-\alpha\right)^{1/2}} - b'\frac{e^2}{4}\int \frac{du}{\mu^2\left(\mu-\alpha\right)^{1/2}}$$

$$= b'\frac{e^2}{2}\frac{1}{\left(1+\beta^2\right)^{1/2}} \tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} - b'\frac{e^2}{4}$$

$$\left\{\frac{(\mu-\alpha)^{1/2}}{\alpha\mu} + \frac{1}{\alpha^{3/2}}\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right\} .$$

Combining terms yields:

$$\Delta s = \frac{b'}{\left(1 + \beta^2\right)^{1/2}} \left\{ \tan^{-1} \left(\frac{\lambda^2 - \beta^2}{1 + \beta^2} \right)^{1/2} \left(1 + \frac{e^2}{2} - \frac{e^2}{4\left(1 + \beta^2\right)} \right) - \frac{e^2}{4\left(1 + \beta^2\right)^{1/2}} \cdot \frac{\left(\lambda^2 - \beta^2\right)^{1/2}}{1 + \lambda^2} \right\} \Bigg|_{\lambda_0}^{\lambda_1} \quad (\lambda \geqslant \beta)$$

$$b' = b\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2} .$$

Now

$$\frac{b'}{\left(1 + \beta^2\right)^{1/2}} = b\left(1 - e^2 \frac{\beta^2}{1 + \beta^2}\right)^{1/2}$$

so that (up to e^2 terms)

$$\Delta s = b\left(1 - \frac{e^2}{2}\frac{\beta^2}{1 + \beta^2}\right) \left\{ \left[\left(1 + \frac{e^2}{2} - \frac{e^2}{4\left(1 + \beta^2\right)}\right) \tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2} \right] - \frac{e^2}{4\left(1 + \beta^2\right)^{1/2}} \cdot \frac{\left(\lambda^2 - \beta^2\right)^{1/2}}{1 + \lambda^2} \right\} .$$

This can be reduced to:

$$\Delta s = b\left\{ \left(1 + \frac{e^2}{4\left(1 + \beta^2\right)}\right) \tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2} - \frac{e^2}{4\left(1 + \beta^2\right)^{1/2}} \cdot \frac{\left(\lambda^2 - \beta^2\right)^{1/2}}{1 + \lambda^2} \right\} \Bigg|_{\lambda_0}^{\lambda_1}$$

Now turn to the task of finding $\vec{D}(\lambda)$ in order to obtain instantaneous heading information.

With,

$$\rho(\lambda) = b\left(1 + \frac{e^2}{2}\frac{\lambda^2}{1 + \lambda^2}\right)$$

$$\phi(\lambda) = \tan^{-1}\lambda$$

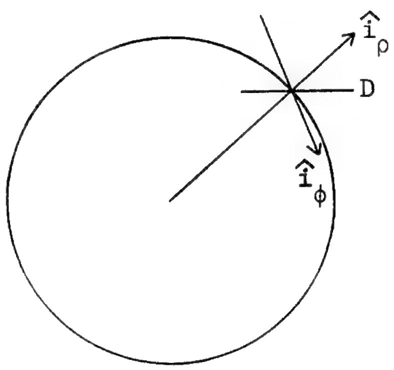
$$\theta(\lambda) = \theta_0 + \sec^{-1}\frac{\lambda}{\beta} - \frac{e^2}{2}\frac{\beta}{\left(1 + \beta^2\right)^{1/2}}\tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2}$$

$$\vec{D}(\lambda) = \frac{d\rho}{d\lambda}\,\hat{i}_\rho + \rho\,\frac{d\phi}{d\lambda}\,\hat{i}_\phi + \rho\,\sin\phi\,\frac{d\theta}{d\lambda}\,\hat{i}_\theta \quad .$$

Eventually the ratio of two components must be evaluated. The vector $\vec{D}(\lambda)$ is divided through by ρ and expanded (up to e^2 only):

$$\vec{D}(\lambda) \cong \left[e^2\,\frac{\lambda}{(1+\lambda^2)^2}\right]\hat{i}_\rho + \left[\frac{1}{1+\lambda^2}\right]\hat{i}_\phi + \left[\frac{1}{(1+\lambda^2)^{1/2}\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}}\left(1-\frac{e^2}{2}\cdot\frac{\lambda^2}{1+\lambda^2}\right)\right]\hat{i}_\theta \quad .$$

Since terms up to e^2 must be retained, note that the $\hat{i}_\rho$ term does not vanish. A look at the geometry shows immediately what must be done next.



NOTE: $\hat{i}_\phi$ does not point along the surface, but into the surface.

Next, a vector $\vec{D}'(\lambda)$ is needed, this vector is normally measured on the surface (ie, $\vec{D}(\lambda)\cdot\hat{i}'_\rho \equiv 0$). What is needed is a rotation about the $\hat{i}_\theta$ axis, the angle is determined by setting $\vec{D}\cdot\hat{i}\rho = 0$, so:

$$\vec{D}'(\lambda) = \begin{pmatrix} \cos\xi & \sin\xi & 0 \\ -\sin\xi & \cos\xi & 0 \\ 0_\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{1+\lambda^2} \\ e^2\lambda\left(1+\lambda^2\right)^{-2} \\ \dfrac{1}{(1+\lambda^2)^{1/2}}\cdot\dfrac{1}{\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}}\left(1-\dfrac{e^2}{2}\,\dfrac{\lambda^2}{1+\lambda^2}\right) \end{pmatrix} .$$

First set:

$$0 = -\sin\xi\left(\frac{1}{1+\lambda^2}\right) + \cos\xi\left(\frac{e^2\lambda}{(1+\lambda^2)^2}\right)$$

$$\tan\xi = \frac{e^2\lambda}{1+\lambda^2} \qquad (0 \leqslant \lambda \leqslant \infty \;\; , \; \xi < \tan^{-1}(e^2) \;\; , \; \xi \text{ bounded and small}).$$

Now, since ξ is very small, let $\cos\xi \sim 1$, $\sin\xi \sim \tan\xi \sim \dfrac{e^2\lambda}{1+\lambda^2}$

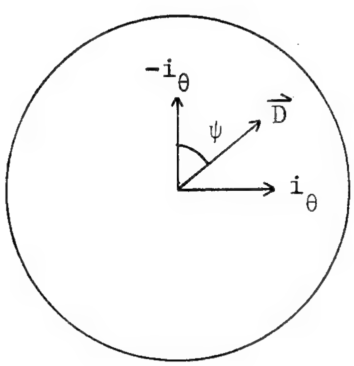
$$\vec{D}'(\lambda) = \frac{\lambda}{1+\lambda^2} + \frac{e^2\lambda}{1+\lambda^2}\left(\frac{e^2\lambda}{(1+\lambda^2)^{1/2}}\right)\hat{i}_\phi' + \frac{1}{(1+\lambda^2)^{1/2}} \cdot \frac{\beta}{(\lambda^2-\beta^2)^{1/2}}$$
$$\left(1 - \frac{e^2}{2}\,\frac{\lambda^2}{1+\lambda^2}\right)\hat{i}_\theta'$$

$$\vec{D}'(\lambda) \cong \frac{1}{1+\lambda^2}\,\hat{i}_\phi' + \frac{\beta}{(1+\lambda^2)^{1/2}(\lambda^2-\beta^2)^{1/2}}\left(1 - \frac{e^2}{2}\,\frac{\lambda^2}{1+\lambda^2}\right)\hat{i}_\theta'$$

(eliminating e^4 terms).

It would appear from this analysis that justification exists for keeping the old $\vec{D}(\lambda)$ and just dropping the heading $\hat{i}_\rho$ term.

Now consider the heading angle ψ. As usual, the work is performed in quadrant I.



$$\tan\psi = \frac{\vec{D}\cdot\hat{i}_\theta}{\vec{D}\cdot i_\phi} = \left(1 - \frac{e^2}{2}\,\frac{\lambda^2}{1+\lambda^2}\right)\beta\,\frac{(\lambda^2+1)^{1/2}}{(\lambda^2-\beta^2)^{1/2}} \quad .$$

(3) Slightly oblate spheroid solutions.

$$\rho(\lambda) = b\left(1 + \frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right)$$

$$\phi(\lambda) = \tan^{-1}\lambda$$

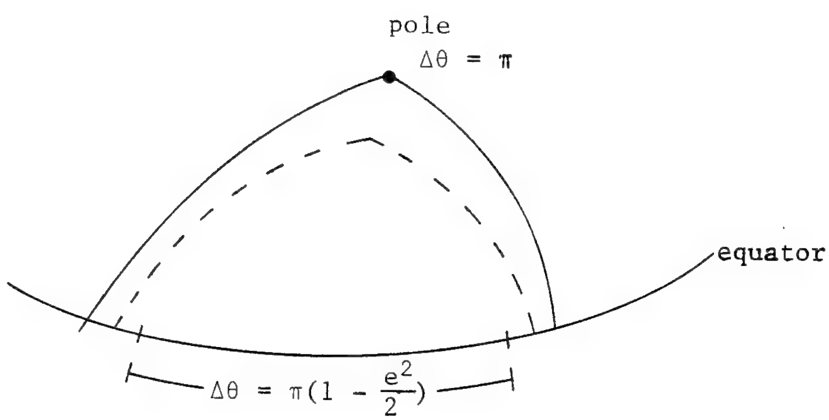
$$\theta(\lambda) = \theta_0 \pm \left\{\sec^{-1}\left(\frac{\lambda}{\beta}\right) - \frac{e^2}{2}\frac{\beta}{(1+\beta^2)^{1/2}}\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right\}$$

$$\Delta s(\lambda) = b\left[\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\left(1 + \frac{e^2}{4(1+\beta^2)}\right) - \frac{e^2}{4(1+\beta^2)}\cdot\frac{(\lambda^2-\beta^2)^{1/2}}{\lambda^2+1}\right]\Bigg|_{\lambda_0}^{\lambda_1}$$

$$\psi(\lambda) = \tan^{-1}\left[\left(1 - \frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right)\beta\left(\frac{\lambda^2+1}{\lambda^2-\beta^2}\right)^{1/2}\right] \qquad (\lambda \geqslant \beta).$$

The most dramatic difference between spherical and oblate spheroid geodesics can be seen by comparing equator to equator trajectories in the oblate transition region.

Consider (northern hemisphere reference only)



For equator to equator trajectories where $\Delta\theta \leqslant \pi\left(1 - \frac{e^2}{2}\right)$ the spherical geodesic is the same as the oblate geodesic.

For equator to equator trajectories where $\pi > \Delta\theta > \pi\left(1 - \frac{e^2}{2}\right)$, the spherical geodesic remains along the equator, whereas the oblate geodesic takes a route similar to that of the dotted line [actually there are two geodesic routes (north and south) by surface symmetry about the equator].

For equator to equator trajectories with $\Delta\theta = \pi$ there are an inifinite number of spherical geodesics, there are only two oblate geodesic routes (one over each pole). In general, when $\vec{\rho}_0 = -\vec{\rho}_1$, for the spherical case there are infinite geodesics, on the oblate surface there may be only two geodesics (over the poles).

Thus, another effect of oblateness is to remove the singularity in the number of geodesics which can occur (exception: for the oblate case there is at least one pair of points which produce infinite geodesics, north pole to south pole).

(4) Comparison of FROST to integrated great circle route. Spherical minima routing gave a relation between ϕ and θ, such that the route was confined to a plane. If this route is integrated on the slightly oblate spheroid, the following equations are obtained:

$$\rho(\lambda) = b\left(1 + \frac{e^2}{2}\,\frac{\lambda^2}{1+\lambda^2}\right)$$

$$\phi(\lambda) = \tan^{-1}\lambda$$

$$\theta(\lambda) = \theta_0 \pm \sec^{-1}\frac{\lambda}{\beta} \quad .$$

Then,

$$\Delta s = b\int\left|1 + \frac{e^2}{2}\,\frac{\lambda^2}{1+\lambda^2}\right|\left|\left(e^4\,\frac{\lambda^2}{(1+\lambda^2)^2} + 1\right)\left(\frac{1}{1+\lambda^2}\right)^2 + \frac{\lambda^2}{1+\lambda^2}\left(\frac{1}{\lambda^2\left(\frac{\lambda^2}{\beta} - 1\right)}\right)\right|^{1/2} d\lambda$$

$$\Delta s = b\left(1+\beta^2\right)^{1/2}\left\{\int\frac{\lambda d\lambda}{(1+\lambda^2)(\lambda^2-\beta^2)^{1/2}} + \frac{e^2}{2}\int\frac{\lambda^2}{(1+\lambda^2)^2}\,\frac{\lambda d\lambda}{(\lambda^2-\beta^2)^{1/2}}\right\}$$

$$\Delta s^* = b\left\{\left(1 + \frac{e^2}{2} - \frac{e^2}{4(1+\beta^2)}\right)\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} - \frac{e^2}{4(1+\lambda^2)}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right\}\Bigg|_{\lambda}$$

and,

*Up to e^2 only.

$$\psi^* = \tan^{-1} \beta \left(\frac{\lambda^2 + 1}{\lambda^2 - \beta^2} \right)^{1/2} \quad .$$

Referring to the geodesy example in Appendix I (New York-Leningrad), the integrated great circle parameters are:

$\beta = 0.46113733$.

$\lambda_a = 1.130294$.

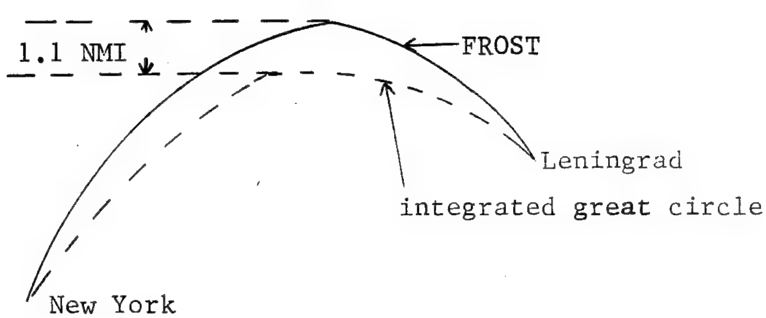
$\lambda_b = 0.577350$.

The appropriate level of significance is $\sim e^4$.

The FROST distance measurement gave $\Delta S \cong 6735.37$ km.

The integrated great circle measurement gives $\Delta S \cong 6735.37$ km.

This result is not surprising if the difference between the two routes is considered:



In both cases the value of β gives the minimum colatitude the route will take. For FROST $\phi_{min} \cong 24.7379°$. For integrated great circle $\phi_{min} \cong 24.7562°$.

The difference between the two comes out to be approximately 1.1 nautical miles.

*Up to e^2 only.

Does this result challenge the validity of FROST? No, rather it certifies it. This particular example has starting and ending points far north of the equatorial bulge (the whole route is far from the equator). It is in cases like this, where the values of λ and β are relatively small that one would expect differences between the two routes to be vanishingly small. On the other hand, routes predominantly close to the equator should show greater distance and routing deviation (consider two points on the equator 179.9° apart in longitude).

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APPENDIX I

10.0 PROJECT FROST

10.1 The need for geodesic and surface navigation algorithms

There exists a continuing military requirement to review and upgrade (as needed) the algorithms used to perform geodesic measurement and perform surface navigation. These algorithms affect the manner in which:

a. Mapping agencies measure distances between points (along a geodesic or curve of minimum distance).

b. Long range aircraft (LRA) navigate on the reference surface.

c. Long range naval operations (surface/submarine/air) perform surface navigation.

d. Long range cruise missile surface guidance is performed during inertial flight phase.

e. Strategic defense initiative surface interceptor problems can be addressed.

(1) Airborne platform positioning and navigation.

(2) Reentry intercept.

(3) Interception of compressed trajectory vehicles via kinetic kill vehicles.

f. Other (NASA type applications).

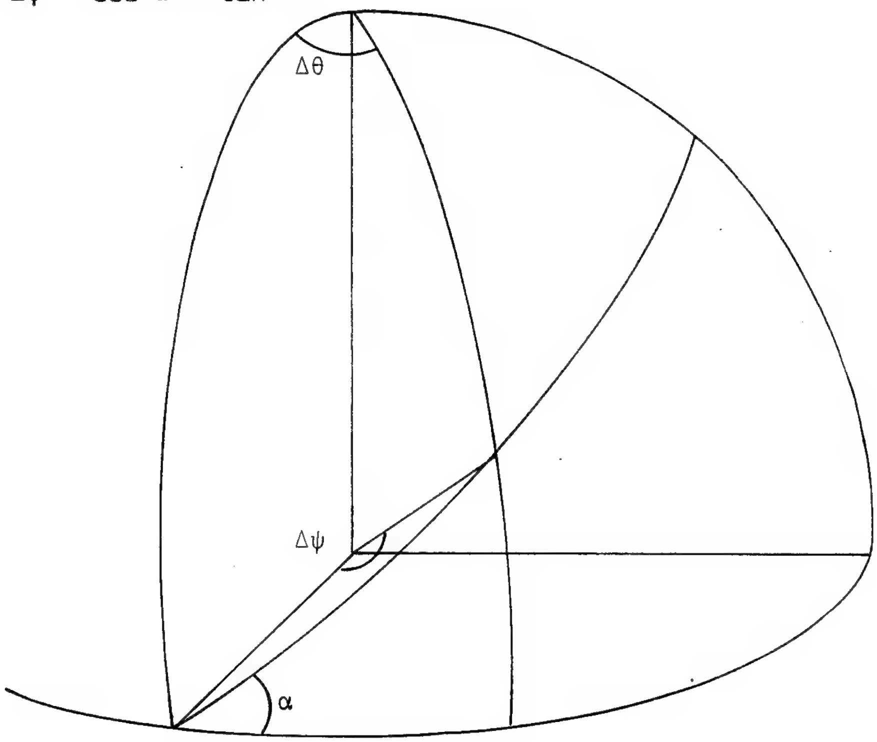
10.2 Current methodology

Use spherical trigonometry, with a compensated equal-area radius.

a. Basic spherical trigonometry relations.

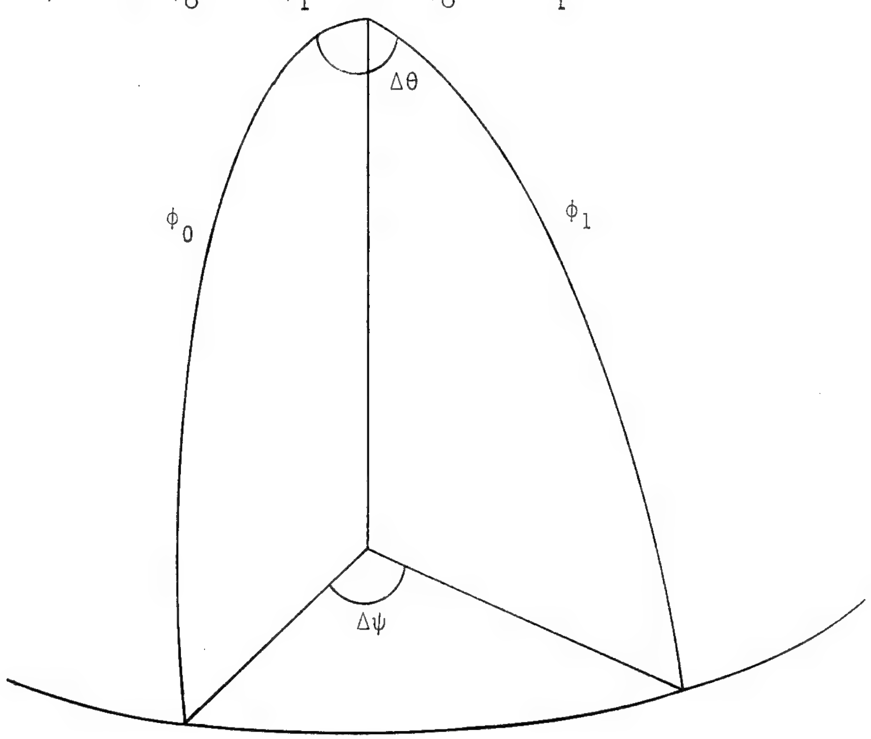
(1) Law of tangents.

$\tan \Delta\psi \cdot \cos \alpha = \tan \Delta\theta$



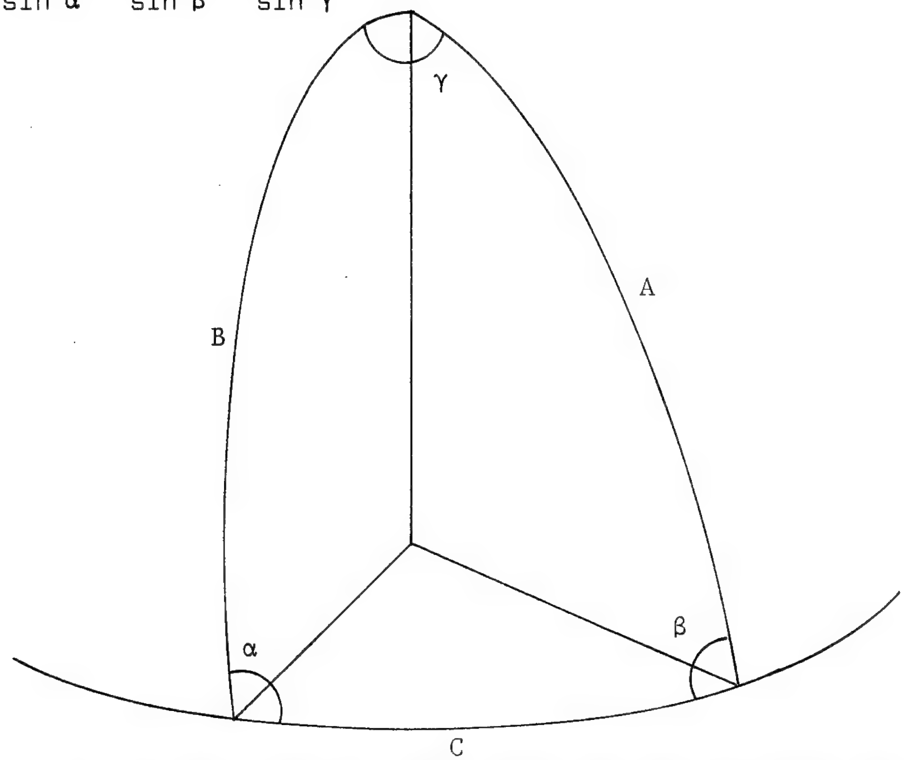
(2) Law of cosines. Law of cosines generally used to measure out distances (can also use tangents relation with proper parameters).

$\cos \Delta\psi = \cos \phi_o \cos \phi_1 + \sin \phi_o \sin \phi_1 \cos \Delta\theta$



(3) Law of sines. Law of sines generally used for heading/directional information.

$$\frac{\sin A}{\sin \alpha} = \frac{\sin B}{\sin \beta} = \frac{\sin C}{\sin \gamma}$$



b. Current method. Uses spherical relationships and equal-area radius value.

(1) The equivalent area sphere to oblate spheriod relation is as follows:

$$4\pi \bar{a}^2 = 2\pi a^2 + \frac{\pi b^2}{e} \cdot \ell n \left|\frac{1 + e}{1 - e}\right| .$$

Rearranging:

$$\bar{a}^2 = \frac{a^2}{2} + \frac{b^2}{4e} \left(\ln |1 + e| - \ln |1 - e| \right)$$

$$\cong \frac{1}{2} \left[a^2 + \frac{b^2}{2e}\left(e - \frac{e^2}{2} + \frac{e^3}{3} + e + \frac{e^2}{2} + \frac{e^3}{3} \right)\right] \quad \text{(assuming } e^2 \text{ small)}$$

$$\cong \frac{1}{2} \left[a^2 + b^2 \left(1 + \frac{e^2}{3} \right)\right]$$

$$\cong \frac{b^2}{2}\left(2 + \frac{4e^2}{3}\right) \qquad \frac{a^2}{b^2} = \frac{1}{1 - e^2} \cong 1 + e^2$$

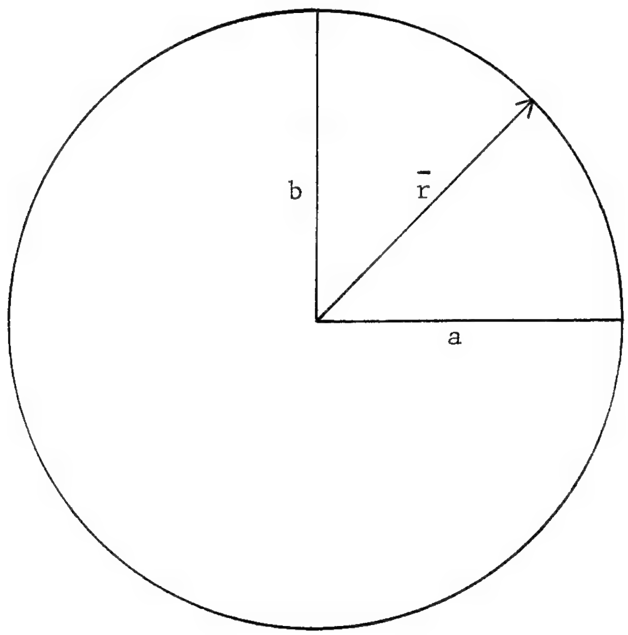
$$\bar{a} \cong b\left(1 + \frac{2e^2}{3}\right)^{1/2}$$

$$\bar{a} \cong b\left(1 + \frac{e^2}{3}\right).$$

(2) The above approximation for the radius value is to be used with Naperian trigonometry (assumed to be valid up to e^2 correction).

10.3 Problems with the current methodology

a. On an oblate spheriod the radius vector magnitude varies from $b \leqslant \left|\vec{r}\right| \leqslant a$.



b. Now since e, the eccentricity can be defined as $e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$, then rearranging

$$a = b(1 - e^2)^{-1/2}$$

and if e << 1 then,

$$a \cong b\left(1 + \frac{e^2}{2}\right),$$

therefore:

$$b \leq \left|\vec{r}\right| \leq b\left(1 + \frac{e^2}{2}\right).$$

c. For the equal area compensation

$$\left|\vec{r}\right| = b\left(1 + \frac{e^2}{3}\right)$$

This value of $\left|\vec{r}\right|$ is assumed to be used over the whole spheriod.

d. A problem arises in the level of accuracy required for precision measurements. The so called equal area method should have an accuracy limitation on the order of (or no less than an order of magnitude less than) e^2 (on the earth, $e^2 \approx .0067$). For quarter turn measurements (the distance swept out by a 90° central angle, this to be redefined later) errors from -5 to +11 kilometers can be obtained.

10.4 Introduction to FROST

a. Start by rederiving the spherical case geodesics, considering each necessary component separately, and relating each component via a parametric variable, λ.

(1) For the spherical case:

$$\rho(\lambda) = a$$

$$\phi(\lambda) = \tan^{-1} \lambda$$

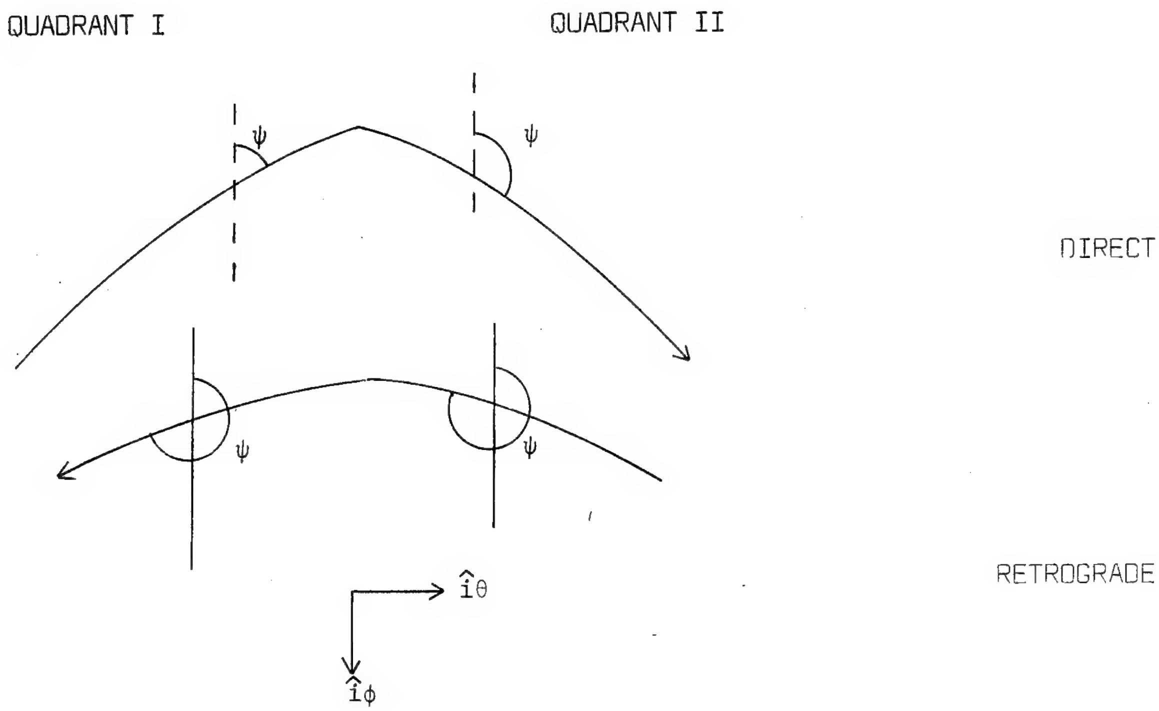
$$\theta(\lambda) = \theta_o \pm \sec^{-1} \frac{\lambda}{\beta}$$

$$\psi(\lambda) = \tan^{-1}\left[\beta\left(\frac{\lambda^2 + 1}{\lambda^2 - \beta^2}\right)^{1/2}\right] \quad \text{(synthetic law of sines)}$$

$$\Delta s(\lambda) = a \tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2}\Bigg|_{\lambda_0}^{\lambda_1} \quad \text{(synthetic law of cosines).}$$

(a) It can be shown that upon elimination of λ, the above relations reduce to standard spherical trigonometry.

(b) Spherical case quadrant analysis for ψ



for ψ the heading or direction angle, $\tan^{-1} \beta \left(\frac{\lambda^2 + 1}{\lambda^2 - \beta^2} \right)^{1/2} = q(\lambda)$ produces the following terms:

Quadrant I	Quadrant II	
$q(\lambda)$	$\pi - q(\lambda)$	DIRECT
$\pi + q(\lambda)$	$2\pi - q(\lambda)$	RETROGRADE

(2) For the slightly oblate spherical case assume the eccentricity value e is small. Then using calculus of variations and solving for the geodesic curves, keeping terms up to e^2 only (eliminating e^4 and higher terms) obtain the following relationships:

$$\rho(\lambda) = b \left(1 + \frac{e^2}{2} \cdot \frac{\lambda^2}{1 + \lambda^2} \right)$$

$$\phi(\lambda) = \tan^{-1} \lambda$$

$$\theta(\lambda) = \theta_o \pm \sec^{-1} \frac{\lambda}{\beta} - \frac{e^2}{2} \cdot \frac{\beta}{\left(1 + \beta^2\right)^{1/2}} \cdot \tan^{-1} \left(\frac{\lambda^2 - \beta^2}{1 + \beta^2} \right)^{1/2}$$

$$\psi(\lambda) = \tan^{-1}\left[\left(1 - \frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right)\beta\left(\frac{\lambda^2+1}{\lambda^2-\beta^2}\right)^{1/2}\right]$$

$$\Delta s(\lambda) = b\left\{\left(1 + \frac{e^2}{4\left(1+\beta^2\right)}\right)\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} - \frac{e^2}{4\left(1+\beta^2\right)^{1/2}}\frac{\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda+1}\right\}\Bigg|_{\lambda_0}^{\lambda_1}$$

(a) ϕ in spherical coordinates is defined as the geocentric colatitude variable. The geodetic latitude variable is related to λ by

$$L(\lambda) = \cot^{-1}\left[(1-e^2)\lambda\right] .$$

(b) In the above equations, in the limit as $e \to o$, the spherical case solutions are recuperated.

b. FROST makes a first order attempt at solving for the geodesics on a slightly oblate surface.

(1) It assumes the surface eccentricity (e) is so small that terms of the order of e^4 and higher may be discarded. This seems a reasonable assumption since:

(a) e^2 corrections are from -5 to +11 kilometers per quarter turn.

(b) e^4 corrections are on the order of tens of meters per quarter turn, assumed to be well within the noise of the actual surface fluctuations of the geoid; therefore,

(c) Higher order terms (greater than e^4) can be absolutely discarded for the earth for any reasonable accuracy.

(2) The above assumptions form the basis for the limitations on the use of FROST.

10.5 Comparison study

a. In comparison with the spherical relations note an apparent crunching up of the curve for θ at the nodal crossings.

$$\theta(\lim \lambda \to \infty) = \theta_o \pm \frac{\pi}{2} \quad \text{sphere}$$

$$\theta(\lim \lambda \to \infty) = \theta_o \pm \frac{\pi}{2}\left(1 - \frac{e^2}{2}\frac{\beta}{\left(1+\beta^2\right)^{1/2}}\right) \text{oblate correction}$$

also, ψ(node) yields:

$$\psi(\lim \lambda \to \infty) = \tan^{-1} \beta \quad \text{sphere}$$

$$\psi(\lim \lambda \to \infty) = \tan^{-1}\left[\left(1 - \frac{e^2}{2}\right)\beta\right] \quad \text{oblate.}$$

NOTE: $\psi(\lim \beta \to \infty) = \frac{\pi}{2}$ in both cases.

$$\Delta s \Big|_{\lambda=\beta}^{\lambda=\infty} = a \cdot \frac{\pi}{2} \quad \text{sphere}$$

$$\Delta s \Big|_{\lambda=\beta}^{\lambda=\infty} = b \frac{\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2}}{\left(1 + \beta^2\right)^{1/2}} \left(1 + \frac{e^2}{2} - \frac{e^2}{4\left(1 + \beta^2\right)}\right) \frac{\pi}{2} \quad \text{oblate.}$$

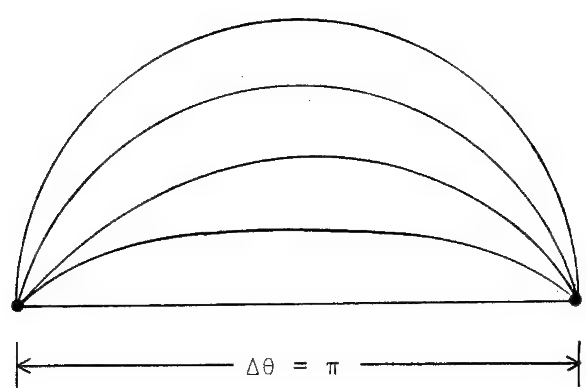
$$\Delta s \Big|_{\lambda=\beta}^{\lambda=\infty} \cong \frac{\pi b}{2}\left(1 - \frac{e^2}{2} \frac{\beta^2}{1 + \beta^2}\right)\left(1 + \frac{e^2}{2} - \frac{e^2}{4\left(1 + \beta^2\right)}\right)$$

$$\cong \frac{\pi b}{2}\left(1 + \frac{e^2}{4} \cdot \frac{1}{1 + \beta^2}\right) \quad \text{(keeping up to } e^2 \text{ terms only).}$$

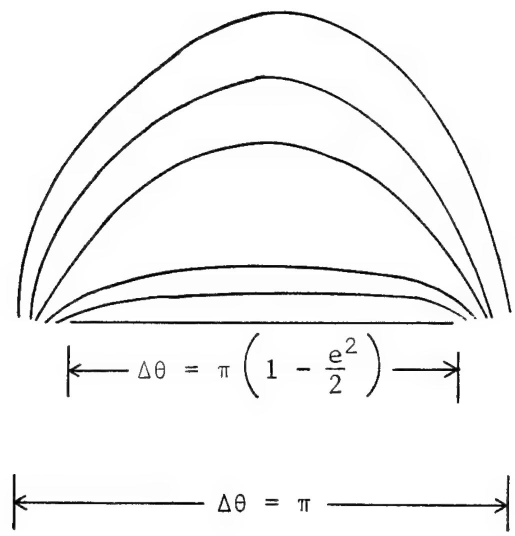
(1) At first glance, this may look like nonsense, as β gets larger, the distance traveled is smaller, but look at $\theta(\lambda)$. As β gets larger $\Delta\theta = \pi$ is not transverse so that this result does make sense.

(2) Parametric geodesics, symmetric about an arbitrary θ_o.

(a) Sphere.



(b) Slightly oblate spheroid:



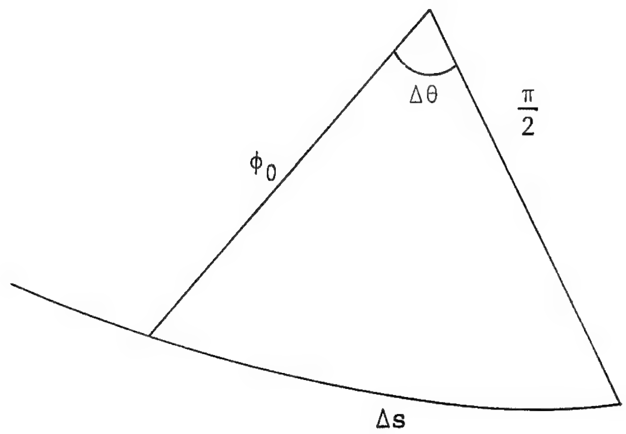
b. Comparison of distance formulas. Oblate spheroid, perturbed spherical routing with great circle, equal area routing. Comparison will be from the high point in the trajectory (or maximum latitude) ($\lambda = \beta$) to the equator ($\lambda = \infty$) so that Δ_s is evaluated as $\beta \leqslant \lambda \leqslant \infty$.

(1) For the perturbed solution,

$$\Delta\theta = \frac{\pi}{2}\left[1 - \frac{e^2}{2}\left(\frac{\beta}{\left(1 + \beta^2\right)^{1/2}}\right)\right]$$

$$\Delta s_n = \frac{\pi b}{2}\left(1 - e^2 \frac{\beta^2}{1 + \beta^2}\right)^{1/2}\left(1 + \frac{e^2}{2} - \frac{e^2}{4\left(1 + \beta^2\right)}\right)\Bigg|_{\lambda=\beta}^{\lambda=\infty} .$$

(2) For the spheroid approximation:



$$\cos\frac{\Delta s}{a} = \cos\phi\,\cos\frac{\pi}{2} + \sin\phi\,\sin\frac{\pi}{2}\cos\Delta\theta$$

$$\Delta s = \bar{a}\cos^{-1}[\sin\phi\,\cos\Delta\theta]$$

but $\cos\Delta\theta = \sin\frac{\pi e^2}{4}\cdot\frac{\beta}{\left(1+\beta^2\right)^{1/2}}$.

It is convenient to call β in these cases $\tan\phi_0$, ϕ_0 the minimum colatitude value for a given geodesic, so that:

$$\Delta s_a = \bar{a}\cos^{-1}\left[\sin\phi_0\cdot\sin\left(\frac{\pi e^2}{4}\sin\phi_0\right)\right] \cong \frac{\pi\bar{a}}{2}\left(1-\frac{e^2}{2}\sin^2\phi_0\right) .$$

$$\Delta s_n = b\,\frac{\pi}{2}\left(1-e^2\sin^2\phi_0\right)^{1/2}\left(1+\frac{e^2}{2}-\frac{e^2}{4}\cos^2\phi_0\right) \cong \frac{\pi b}{2}\left(1+\frac{e^2}{4}\cos^2\phi_0\right)$$

(up to e^2 terms).

Now, with $e^2 \cong .0067227$,

$\bar{a}$ = 6371.227 km

b = 6356.912 km .

(3) For the quarter turn (defined as $\beta \leq \lambda \leq \infty$):

ϕ_o	Δs_n	Δs_a
0	10002.196	10007.899
10	10001.690	10006.885
20	10000.233	10003.964
30	9998.000	9999.489
45	9993.805	9991.079
60	9989.609	9982.669
75	9986.538	9976.513
90	9985.414	9974.259

All figures in kilometers.

$\Delta s_n = \Delta s_a$ at $\phi_o = 35.568°$

Max difference $|\Delta s_n - \Delta s_a|$ positive = 11.155 km

$|\Delta s_n - \Delta s_a|$ negative = 5.703 km

The approximation is good to two significant digits.

c. The comparison of FROST versus spherical trigonometry is shown in figure 1.

10.6 Geodesic measurement example

a. Measure distance between two theoretical bases:

(1) Base A: 41°.5000 N, 73°.1146 W (New York) geocentric latitude.

(2) Base B: 60°.0000 N, 29°.8000 E (Leningrad) geocentric latitude.

(3) Δlongitude = 102°.9146

(a) 41°.6918 N geodetic latitude.

(b) 60°.1670 N geodetic latitude.

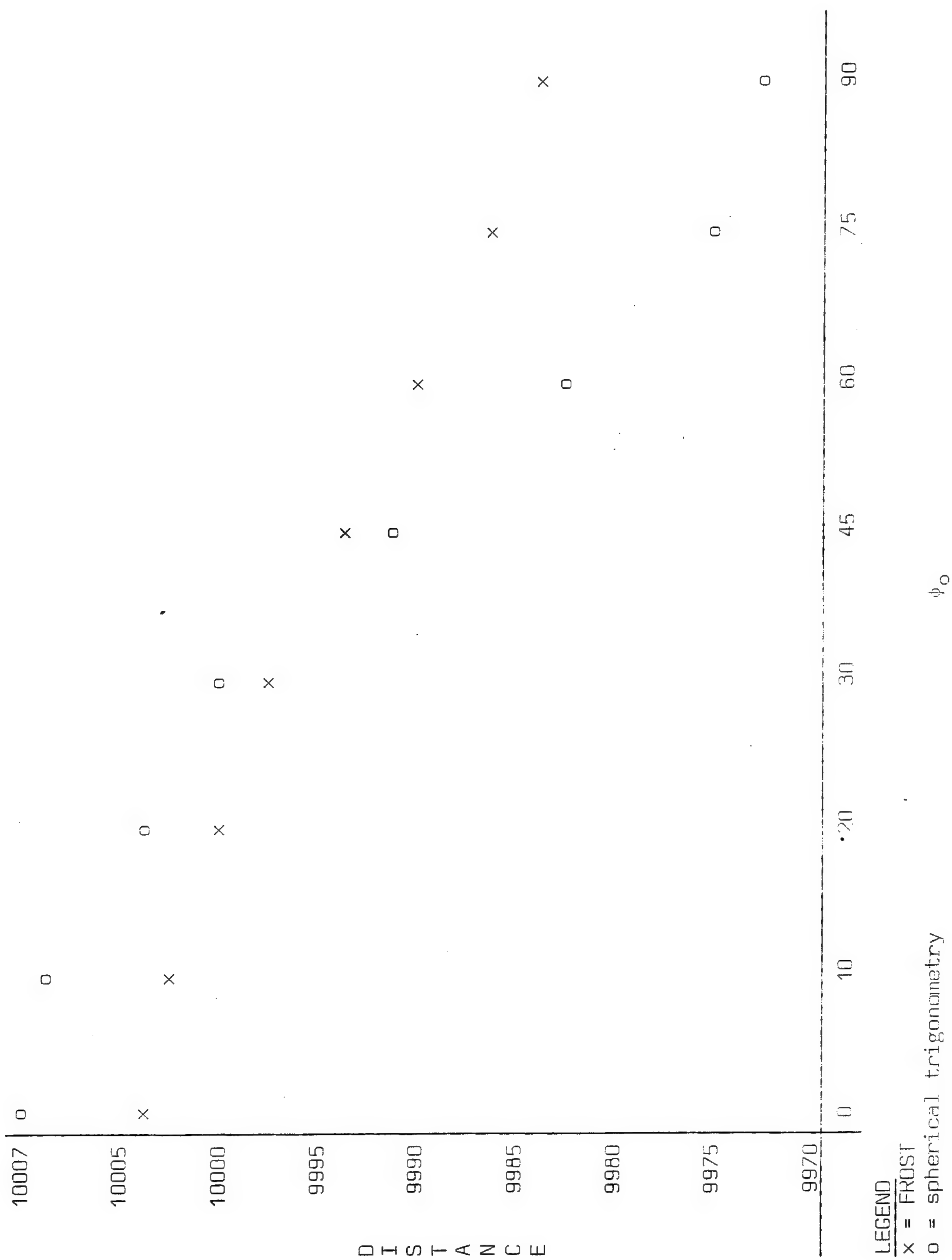


Figure 10.1 Measurement comparison for a quarter turn

b. Comparison of distances measured.

(1) Spherical trigonometry, equal area e^2 compensation, geocentric.

Δs = 6744.0840 km.

(2) Spherical trigonometry, equal area e^2 compensation, geodetic.

Δs = 6716.2297 km. $\Delta \cong 8.7$km

(3) FROST (β = .460749734, θ_o = 7°2318). $\Delta \cong 19.1$km

Δs = 6735.3776 km.

(a) Spherical trigonometry w/e^2 compensation, geocentric.

Let,

A be at longitude o°

B at longitude 102°9146 .

$A = \{.748955721,\ 0,\ .662620048\}$

$B = \{-.111749248,\ .487352137,\ .866025404\}$

$\Theta = 60°64952234$

$$\Delta s = b\left(1 + \frac{e^2}{3}\right)(1.058533854)$$

$$= 6744.0840 \text{ km.}$$

NOTE: $\Theta = \cos^{-1}(\vec{A} \cdot \vec{B})$

(b) Spherical trigonometry w/e^2 compensation, geodetic.

$A = \{.746732702,\ 0,\ .665124253\}$

$B = \{-.111184443,\ .484888955,\ .867479522\}$

$\Theta = 60°39902851$

$$\Delta s = b\left(1 + \frac{e^2}{3}\right)(1.054161913)$$

$$= 6716.2297 \text{ km.}$$

NOTE: $\Theta = \cos^{-1}(\vec{A} \cdot \vec{B})$

(c) FROST calculation ($\theta_0 = 7\overset{\circ}{.}2318$, $\beta = .460749734$)

A, $\lambda_A = 1.130294$, $\lambda_A^2 = 1.277565$

B, $\lambda_B = 0.577350$, $\lambda_B^2 = 0.333333$

$$\Delta s_a = b \tan^{-1}\left(\frac{1.277565 - 0.2122903}{1.2122903}\right)^{1/2}\left(1 + \frac{0.006722}{4(1.2122903)}\right)$$

$$- \frac{0.006722}{4\left(1.2122903\right)^{1/2}} \cdot \frac{\left(1.277565 - .2122903\right)^{1/2}}{2.277565}$$

$$\Delta s_b = b \tan^{-1}\left(\frac{.333333 - 0.2122903}{1.2122903}\right)^{1/2}\left(1 + \frac{0.006722}{4(1.2122903)}\right)$$

$$- \frac{0.006722}{4\left(1.2122903\right)^{1/2}} \cdot \frac{\left(.33333 - .2122903\right)^{1/2}}{1.33333}$$

$\Delta s_a = b\{.754145026 - .000691664\} = b(.753453361)$

$\Delta s_b = b\{.306480974 - .0003982597\} = b(.306082714)$

$\Delta s_T = b(1.059536075)$, with $b = 6356.912$

$= 6735.3776$.

Given,

$\beta = .460749734$

$\lambda_a = 1.130294$

$\lambda_b = 0.577350$

calculate $\Delta\theta_T$:

$$\Delta\theta = \sec^{-1}\frac{\lambda}{\beta} - \frac{e^2}{2}\frac{\beta}{\left(1 + \beta^2\right)^{1/2}} \tan^{-1}\left(\frac{\lambda^2 - \beta^2}{1 + \beta^2}\right)^{1/2}$$

$$\Delta\theta_L = 65.94351463 - .06068861 \cong 65.882826$$

$$\Delta\theta_R = 37.05647301 - .024663507 = 37.031809$$

$$\Delta\theta_T = 102.9146(35) \quad .$$

APPENDIX II

20.0 THE TORUS

The torus is a very complicated, multiple connected surface, and, as a result, one should not expect nice closed form solutions.

What is presented here is just an outline of how one might go about solving for the geodesics on such a surface. The solutions presented are strictly formal.

Define the surface model:

$$(r - a)^2 + z^2 = b^2 \qquad (a > b).$$

Then let,

$$(r - a)^2 = \frac{b^2}{1 + \lambda^2}$$

$$r = a \pm \frac{b}{(1 + \lambda^2)^{1/2}}$$

$$z = \pm \frac{b\lambda}{(1 + \lambda^2)^{1/2}}$$

$$s = \int \left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\theta}{d\lambda}\right)^2 \right]^{1/2} d\lambda$$

and

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{b^2\lambda^2}{(1 + \lambda^2)^3}$$

$$\left(\frac{dz}{d\lambda}\right)^2 = \frac{b^2}{(1 + \lambda^2)^3}$$

$$s = a \int \left[\frac{\varepsilon^2}{(1 + \lambda^2)^2} + \hat{r}^2 \left(\frac{d\theta}{d\lambda}\right)^2 \right]^{1/2} d\lambda \quad \left(\varepsilon = \frac{b}{a} \;,\; \hat{r} = 1 \pm \frac{\varepsilon}{(1 + \lambda^2)^{1/2}} \;,\right.$$

$$\left. g = \theta \;,\; g' = \frac{d\theta}{d\lambda} \right).$$

Since

$$\frac{\partial \delta}{\partial g} = 0$$

$$\frac{\hat{r}^2 g'}{\left[\frac{\varepsilon^2}{\left(1 + \lambda^2\right)^2} + \hat{r}^2 \; g'^2\right]^{1/2}} = c$$

and

$$\hat{r}^4 g'^2 = \frac{c^2 \varepsilon^2}{\left(1 + \lambda^2\right)^2} + r^2 c^2 g'^2$$

$$g' = \pm \frac{c\varepsilon}{1 + \lambda^2} \cdot \frac{1}{\hat{r}} \cdot \frac{1}{\left(\hat{r}^2 - c^2\right)^{1/2}} \qquad (|\hat{r}| > |c|) \; .$$

$$\theta(\lambda) = \theta_0 \pm c\varepsilon \int \frac{d\lambda}{\left(1 + \lambda^2\right) \hat{r} \left(\hat{r}^2 - c^2\right)^{1/2}} \; .$$

Continuing,

$$\Delta s = a \int \left[\frac{\varepsilon^2}{\left(1 + \lambda^2\right)^2} + \frac{c^2 \varepsilon^2}{\left(1 + \lambda^2\right)^2} \cdot \frac{1}{\left(r^2 - c^2\right)}\right]^{1/2} d\lambda$$

$$\Delta s = a \int \frac{\varepsilon}{1 + \lambda^2} \left[\frac{\hat{r}^2}{\hat{r}^2 - c^2}\right]^{1/2} d\lambda \; .$$

Finally,

$$\vec{D} = \frac{dz}{d\lambda} \hat{i}_z + \frac{dr}{d\lambda} \hat{i}_r + r \frac{d\theta}{d\lambda} \hat{i}_\theta$$

$$\vec{D}(\lambda) = \pm \left[\frac{b}{\left(1 + \lambda^2\right)^{3/2}}\right] \hat{i}_z \pm \left[\frac{b\lambda}{\left(1 + \lambda^2\right)^{3/2}}\right] \hat{i}_r + \left[a \pm b\left(1 + \lambda^2\right)^{-2} \frac{(\pm)c\varepsilon}{1 + \lambda^2} \cdot \frac{1}{\hat{r}\left(\hat{r} - c^2\right)^{1/2}}\right] \hat{i}_\theta$$

$$\vec{D}(\lambda) = \pm \hat{i}_z \pm \lambda \hat{i}_r \pm \left[\frac{\left(1 + \lambda^2\right)^{1/2}}{\varepsilon} \pm 1 \quad \frac{c\varepsilon}{\hat{r}\left(\hat{r}^2 - c^2\right)^{1\ 2}}\right] \hat{i}_\theta \; .$$

This can be simplified somewhat by letting $\lambda = \tan \psi$, then

$r = a \pm b \cos \psi$

$z = \pm b \sin \psi$

$$\theta(\psi) = \theta_0 \pm c\varepsilon \int \frac{d\psi}{\hat{r}\left(\hat{r}^2 - c^2\right)^{1\ 2}} \qquad \left(\hat{r} = 1 \pm \varepsilon \cos\psi\right)$$

$$\Delta s(\psi) = b \int \frac{\hat{r}}{\left(\hat{r}^2 - c^2\right)^{1/2}}\, d\psi$$

$$\vec{D}(\psi) = \pm\ \hat{i}_z \pm \tan\psi \hat{i}_r \pm \left[\left(\frac{\sec\psi}{\varepsilon} \mp 1\right) \cdot \frac{c\varepsilon}{\hat{r}\left(\hat{r}^2 - c^2\right)^{1/2}}\right] \hat{i}_\theta \quad .$$

APPENDIX III

30.0 A STUDY ON SURFACE CURVATURE

Surface curvature may be evaluated using the parametric geodesics and the appropriate Frenet-Serrat equations because the geodesics for a given surface contain within them qualitative as well as quantitative information regarding the surface itself. The parametric geodesics form a natural network or grid which define the surface.

This evaluation is started by presenting the Frenet-Serrat equation for curvature, evaluating the Christoffel symbol of the second kind, showing the parametric evaluation for the symbol, then proceeding directly to the spherical case.

If x^i are the coordinate variables, $\lambda^i = \frac{dx^i}{ds}$ the unit tangent, then

$$\frac{d\lambda^i}{ds} + \Gamma^i_{\alpha,\beta} \lambda^\alpha \frac{dx^\beta}{ds} = K\mu^i$$ (where K is the curvature, μ^i is a unit vector).

Now,

$$\Gamma^1_{2,2} = -r \ ,$$

$$\Gamma^1_{3,3} = -r \sin^2 \phi \ ,$$

$$\Gamma^2_{1,2} = \frac{1}{r} \ ,$$

$$\Gamma^2_{3,3} = -\sin \phi \cos \phi \ ,$$

$$\Gamma^3_{1,2} = \frac{1}{r} \ ,$$

$$\Gamma^3_{2,3} = \cot \phi \ .$$ All other $\Gamma^i_{\alpha,\beta} \equiv 0$.

For the sphere (r = a) and ,

$$\Gamma^1_{2,2} = -a \ ,$$

$$\Gamma^1_{3,3} = -a \frac{\lambda^2}{1 + \lambda^2}$$

$$\Gamma^2_{1,2} = \frac{1}{a} \quad ,$$

$$\Gamma^2_{3,3} = - \frac{\lambda}{1 + \lambda^2} \quad ,$$

$$\Gamma^3_{1,3} = \frac{1}{a} \quad ,$$

$$\Gamma^3_{2,3} = \frac{1}{\lambda} \quad .$$

30.1 Spherical case

Letting

$x^1 = a$

$x^2 = \tan^{-1} \lambda$

$x^3 = \theta_0 + \sec^{-1} \frac{\lambda}{\beta}$ (choosing the positive side).

NOTE: The result is the same if $x^3 = \theta_0 - \sec^{-1} \frac{\lambda}{\beta}$ is choosen.

Then,

$\lambda^1 = 0$

$$\lambda^2 = \frac{1}{1 + \lambda^2} \frac{d\lambda}{ds}$$

$$\lambda^3 = \frac{1}{\lambda \left(\frac{\lambda^2}{\beta^2} - 1 \right)^{1/2}} \frac{d\lambda}{ds}$$

and with

$$ds = d\lambda\left[a\left(1+\beta^2\right)^{1/2}\frac{\lambda}{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}\right]$$

$$\frac{d\lambda}{ds} = \frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{a\left(1+\beta^2\right)^{1/2}\lambda} \quad ,$$

$$\lambda^2 = \frac{\left(\lambda^2-\beta^2\right)^{1/2}}{a\left(1+\beta^2\right)^{1/2}\lambda}$$

$$\lambda^3 = \frac{\beta}{a\left(1+\beta^2\right)^{1/2}}\frac{1+\lambda^2}{\lambda^2} = \left[\frac{\beta}{a\left(1+\lambda^2\right)^{1/2}}\left(1+\frac{1}{\lambda^2}\right)\right]$$

$$\frac{d\lambda^1}{ds} = 0$$

and

$$K\mu^1 = 0 - a\left[\frac{\lambda^2-\beta^2}{a^2\left(1+\beta^2\right)\lambda^2}\right] - a\,\frac{\lambda^2}{1+\lambda^2}\left[\frac{\beta^2}{a^2\left(1+\beta^2\right)}\cdot\frac{\left(1+\lambda^2\right)^2}{\lambda^2}\right]$$

$$= -\frac{1}{a\left(1+\beta^2\right)}\left[\frac{\lambda^2-\beta^2}{\lambda^2}+\frac{\beta^2\left(1+\lambda^2\right)}{\lambda^2}\right]$$

$$= -\frac{1}{a\left(1+\beta^2\right)}\frac{1}{\lambda^2}\left[\lambda^2-\beta^2+\beta^2+\beta^2\lambda^2\right]$$

$$= -\frac{1}{a} \quad .$$

Evaluating,

$$\frac{d\lambda^2}{ds} = \frac{d}{d\lambda}\left(\frac{\left(\lambda^2-\beta^2\right)^{1/2}}{a\left(1+\beta^2\right)^{1/2}\lambda}\right)\frac{d\lambda}{ds}$$

$$= \frac{1}{a\left(1+\beta^2\right)^{1/2}}\left[\frac{\lambda^2\left(\lambda^2-\beta^2\right)^{-1/2}-\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda^2}\right]\frac{1}{a\left(1+\beta^2\right)^{1/2}}$$

$$\cdot\frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda}$$

$$= \frac{1+\lambda^2}{a^2\left(1+\beta^2\right)}\cdot\frac{1}{\lambda^3}\left[\lambda^2-\left(\lambda^2-\beta^2\right)\right]$$

$$= \frac{\beta^2\left(1 + \lambda^2\right)}{a^2\left(1 + \beta^2\right)\lambda^3} \quad .$$

Now

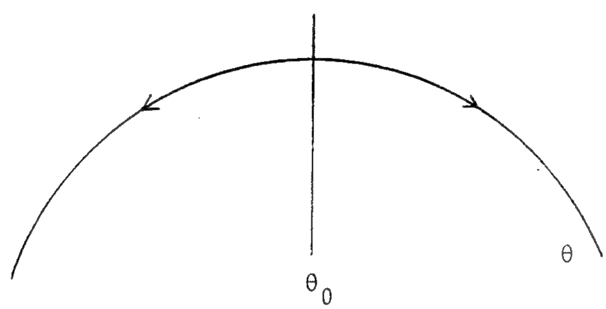
$$K\mu^2 = \frac{\beta^2\left(1 + \lambda^2\right)}{a^2\left(1 + \beta^2\right)\lambda^3} + \frac{1}{a}\left[\lambda^1\lambda^2\right] - \frac{\lambda}{1 + \lambda^2}\left[\lambda^3\right]^2$$

$$= \frac{\beta^2\left(1 + \lambda^2\right)}{a^2\left(1 + \beta^2\right)\lambda^3} + 0 - \frac{\lambda}{1 + \lambda^2}\left|\frac{\beta^2}{a^2\left(1 + \beta^2\right)} \cdot \frac{\left(1 + \lambda^2\right)^2}{\lambda^4}\right|$$

$$= 0 \quad .$$

Evaluating $\frac{d\lambda^3}{ds}$ is somewhat more difficult. Remember that the great circle route in parametric curvilinear coordinates is a multiple defined curve $\theta = \theta_0 \pm \sec^{-1}\frac{\lambda}{\beta}$, based on a multivalved function.

Consider:



The first derivative is $\pm\frac{d\theta}{d\lambda}$, depending on which side of the curve (or, better said, which branch) is chosen for use; however, the second derivative $\frac{d^2\theta}{d\lambda^2}$ is both equal in magnitude and direction (all concave down) on either side, so that a straight forward brute force $\frac{d^2\theta}{d\lambda^2}$ evaluation gives contributions from both branches. The term must be divided by two for use in the curvature equation.

Continuing with,

$$\frac{d\lambda^3}{ds} = \left(\frac{1}{2}\right)\frac{d}{d\lambda}\left(\frac{\beta}{a\left(1+\beta^2\right)^{1/2}} \cdot \frac{1+\lambda^2}{\lambda^2}\right)\frac{1}{a\left(1+\beta^2\right)^{1/2}} \cdot \frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda}$$

$$= \left(\frac{1}{2}\right)\frac{\beta}{a^2\left(1+\beta^2\right)}\left(\frac{2\lambda^3 - 2\lambda\left(1+\lambda^2\right)}{\lambda^4}\right)\frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda}$$

$$= -\frac{\beta}{a^2\left(1+\beta^2\right)} \cdot \frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda^4}$$

and

$$K\mu^3 = -\frac{\beta}{a^2\left(1+\beta^2\right)} \cdot \frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda^4} + \frac{1}{a}\left[\lambda^1\lambda^3\right] + \frac{1}{\lambda}\left[\lambda^2\lambda^3\right]$$

$$= -\frac{\beta}{a^2\left(1+\beta^2\right)} \cdot \frac{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda^4} + 0 + \frac{1}{\lambda}\left(\frac{\left(\lambda^2-\beta^2\right)^{1/2}}{a\left(1+\beta^2\right)^{1/2}\lambda}\right)$$

$$\left(\frac{\beta}{a\left(1+\beta^2\right)^{1/2}}\frac{1+\lambda^2}{\lambda^2}\right)$$

$$= 0 \ .$$

So, the solution for $K\mu^i = \frac{1}{a}\begin{pmatrix}-1\\0\\0\end{pmatrix}$, constant and in the $-\hat{i}_0$ direction. This is known to be generally true, not only for the great circle routes but for the surface of a sphere [constant curvature, independent of where the observer is (θ_0, λ) or in what direction the observer is looking (β)]. This is an important result. It also points out what would have happened had not 1/2 of the $\frac{d\lambda^3}{ds}$ term been taken when evaluating the curvature ($K\mu^3 \neq 0$ but a function of λ, β which would not make sense).

For completeness lets turn our attention to evaluating the torsion of great circle routes.

Fortunately it is a trivial matter to show that τ, the torsion for the parametric spherical geodesic is $\tau \equiv 0$.

From

$$\frac{d\mu^i}{ds} + \Gamma^i_{\alpha\beta}\,\mu^\alpha\,\frac{dx^\beta}{ds} = -\left(K\lambda^i + \tau\mu^i\right)$$

now

$$\frac{dx^\beta}{ds} = \lambda^\beta$$

and

$$K\mu^i = \frac{1}{a}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \qquad K = \frac{1}{a} \quad , \ \mu = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{d\mu}{ds} \equiv 0 \quad .$$

What is needed to be shown is:

$$\Gamma^i_{\alpha\beta}\, \mu^\alpha \lambda^\beta + K\lambda^i = 0 = \tau\mu^i \quad .$$

NOTE: $\alpha \equiv 1$ always, otherwise $\mu = 0$.

For $i = 1$,

$$\Gamma^1_{1,\beta} = 0$$

$$\lambda^1 = 0$$

$$\tau\mu^1 = 0 \quad .$$

For $i = 2$,

$$\Gamma^2_{1,2} = \frac{1}{a}$$

so that

$$\left(\frac{1}{a}\right)\left(-1\right)\lambda^2 + \frac{1}{a}\lambda^2 = 0 \quad ,$$

$$\tau\mu^2 = 0 \quad .$$

For $i = 3$,

$$\Gamma^3_{1,3} = \left(\frac{1}{a}\right)\left(-1\right)\lambda^3 + \frac{1}{a}\lambda^3 = 0 \quad ,$$

$\tau\mu^3 = 0$

QED.

For the minima solutions on the sphere there is zero torsion and constant curvature, independent of β or λ. It is probably primarily for this reason that the sphere is so clean to work on.

The end of this section will concern itself with the evaluation of curvature on a slightly oblate planet. Coordinates to be used are, of course, spherical.

The appropriate expressions are:

$$\rho = b\left(1 + \frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right) = x^1$$

$$\phi = \tan^{-1}\lambda = x^2$$

$$\theta = \theta_0 + \sec^{-1}\frac{\lambda}{\beta} - \frac{e^2}{2}\frac{\beta}{\left(1+\beta^2\right)^{1/2}}\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} = x^3 \quad .$$

Now,

$$\frac{ds}{d\lambda} = b\left(1+\left(1-e^2\right)\beta^2\right)^{1/2}\left(1+\frac{e^2}{2}\frac{\lambda^2}{1+\lambda^2}\right)\left(\frac{\lambda}{\left(1+\lambda^2\right)\left(\lambda^2-\beta^2\right)^{1/2}}\right)$$

and,

$$\frac{d\rho}{d\lambda} = \frac{be^2}{\left(1+\lambda^2\right)^2}\lambda$$

$$\frac{d\phi}{d\lambda} = \frac{1}{1+\lambda^2}$$

$$\frac{d\theta}{d\lambda} = \frac{1}{\lambda\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}} - \frac{e^2}{2}\frac{\lambda}{\left(1+\lambda^2\right)\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}}$$ (in the derivation e^{4+}

terms will be collected and ignored when justified).

$$\frac{d\lambda}{ds} = \frac{1}{b\left(1+\left(1-e^2\right)\beta^2\right)^{1/2}}\cdot\frac{\left(1+\lambda^2\right)^2}{1+\left(1+\frac{e^2}{2}\right)\lambda^2}\cdot\frac{\left(\lambda^2-\beta^2\right)^{1/2}}{\lambda}$$

$$\lambda^1 = \frac{d\rho}{ds} = e^2 \frac{\left(\lambda^2 - \beta^2\right)^{1/2}}{\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2}} \cdot \frac{1}{1 + \left(1 + \frac{e^2}{2}\right)\lambda^2} = \frac{dx^1}{ds}$$

$$\lambda^2 = \frac{d\phi}{ds} = \frac{\left(1 + \lambda^2\right)\left(\lambda^2 - \beta^2\right)^{1/2}}{b\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2}\left[1 + \left(1 + \frac{e^2}{2}\right)\lambda^2\right]} \frac{1}{\lambda} = \frac{dx^2}{ds}$$

$$\lambda^3 = \frac{d\theta}{ds} = \frac{\beta}{b\left(1 + \left(1 - e^2\right)\beta^2\right)^{1/2}} \cdot \frac{\left(1 + \lambda^2\right)}{\left[1 + \left(1 + \frac{e^2}{2}\right)\lambda^2\right]}\left[\frac{\left(1 + \lambda^2\right)}{\lambda^2} - \frac{e^2}{2}\right] = \frac{dx^3}{ds} \; .$$

Now, the $\frac{d\lambda^i}{ds}$ terms need to be formed, remembering to take $\frac{1}{2}\frac{d}{ds}\left(\frac{d\theta}{ds}\right)$.

After considerable algebra, and keeping terms up to e^2 only:

$$\frac{d\lambda^1}{ds} \cong \frac{e^2\left(1 + \lambda^2\right)^2\left(1 - \lambda^2 + 2\beta^2\right)}{b\left(1 + \left(1 - e^2\right)\beta^2\right)\left[1 + \left(1 + \frac{e^2}{2}\right)\lambda^2\right]^3}$$

$$\frac{d\lambda^2}{ds} = \frac{\beta^2\left(1 + \lambda^2\right)^4}{b^2\left(1 + \left(1 - e^2\right)\beta^2\right)\lambda^3\left[1 + \left(1 + \frac{e^2}{2}\right)\lambda^2\right]^3}$$

$$+ e^2 \frac{\lambda^2\left(1 + \lambda^2\right)^2 \left|3\beta^2 + \lambda^2\left(\beta^2 - 3\right)\right|}{2b^2\left(1 + \left(1 - e^2\right)\beta^2\right)\lambda^3\left[1 + \left(1 + \frac{e^2}{2}\right)\lambda^2\right]^3}$$

and

$$\frac{d\lambda^3}{ds} = \frac{-\beta\left(1 + \lambda^2\right)^2\left(\lambda^2 - \beta^2\right)^{1/2}}{b^2\left(1 + \left(1 - e^2\right)\beta^2\right)\left[1 + \left(1 + \frac{e^2}{2}\right)\lambda^2\right]^3}\left[\frac{e^2}{2} + \frac{\left(1 + \lambda^2\right)^2 + \frac{e^2}{2}\lambda^2\left(2 + \lambda^2\right)}{\lambda^4}\right] .$$

Finally the curvature can be calculated:

$$\frac{d\lambda^i}{ds} + \Gamma^i_{\alpha,\beta}\,\lambda^\alpha\lambda^\beta = K\mu^i \; .$$

Again, after considerable algebra, the following is obtained:

$$K\mu^1 = -\frac{1}{b} + \frac{e^2}{b}\left[\frac{1}{2}\frac{\lambda^2}{1 + \lambda^2} + \frac{\left(1 + 2\beta^2 - \lambda^2\right)}{\left(1 + \beta^2\right)\left(1 + \lambda^2\right)}\right]$$

$$K\mu^2 = \frac{e^2}{b^2\left(1+\lambda^2\right)\lambda} \cdot \frac{\left(3\beta^2 + 2\lambda^2\beta^2 - 2\lambda^2\right)}{\left(1+\beta^2\right)}$$

$$K\mu^3 \equiv 0 \ .$$

Combining all terms, the curvature can be written as:

$$K_0\mu_0^i + K_1\mu_1^i$$

where $K_1\mu_1^i$ is an e^2 perturbing term.

$$K_0\mu_0^i = \frac{1}{b}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$K_1\mu_1^i = e^2\begin{pmatrix} \dfrac{1}{b}\left(\dfrac{\lambda^2}{2\left(1+\lambda^2\right)} + \dfrac{1+2\beta^2-\lambda^2}{\left(1+\beta^2\right)\left(1+\lambda^2\right)}\right) \\ \dfrac{3\beta^2 + 2\lambda^2\beta^2 - 2\lambda^2}{b^2\left(1+\beta^2\right)\left(1+\lambda^2\right)\lambda} \\ 0 \end{pmatrix}$$

$$|K| = \left(g_{ij}\left(K\mu^i\right)\left(K\mu^j\right)\right)^{1/2} \ .$$

30.2 Conic curvature

$$\rho = a\lambda$$

$$\phi = \sin^{-1}\alpha$$

$\theta = \theta_0 \pm \frac{1}{\alpha}\sec^{-1}\frac{\lambda\alpha}{c}$ (the positive branch is chosen and as in the case of the sphere take one-half the value of $\frac{d^2\theta}{ds^2}$).

$$\frac{d\rho}{d\lambda} = a$$

$$\frac{d\phi}{d\lambda} = 0$$

$$\frac{d\theta}{d\lambda} = \frac{1}{\alpha\lambda}\frac{1}{\left(\dfrac{\alpha^2\lambda^2}{c^2} - 1\right)^{1/2}} = \frac{c}{\alpha\lambda\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}$$

$$\frac{ds}{d\lambda} = \frac{a\,\alpha\,\lambda}{\left(\alpha^2\lambda^2 - c^2\right)^{1/2}} \quad .$$

Now,

$$\frac{d\rho}{ds} = \frac{\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{\alpha\lambda}$$

$$\frac{d\phi}{ds} = 0$$

$$\frac{d\theta}{ds} = \frac{c}{a\,\alpha^2\lambda^2}$$

$$\frac{d^2\rho}{ds^2} = \frac{1}{\alpha^2\lambda^2}\left[\alpha\lambda\left(\alpha^2\lambda^2 - c^2\right)^{-1/2}\alpha^2\lambda - \left(\alpha^2\lambda^2 - c^2\right)^{1/2}\alpha\right] \cdot \frac{\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a\,\alpha\,\lambda}$$

$$= \frac{1}{\alpha\lambda^2\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}\left[\alpha^2\lambda^2 - \left(\alpha^2\lambda^2 - c^2\right)\right] \cdot \frac{\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a\,\alpha\,\lambda}$$

$$= \frac{c^2}{\alpha\lambda^2\left(\alpha^2\lambda^2 - c^2\right)^{1/2}} \cdot \frac{\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a\,\alpha\,\lambda}$$

$$= \frac{c^2}{a\,\alpha^2\lambda^3}$$

$$\frac{d^2\phi}{ds^2} \equiv 0$$

$$\frac{1}{2}\frac{d^2\theta}{ds^2} = -\frac{c}{a\,\alpha^2\lambda^3} \cdot \frac{\alpha^2\lambda^2 - c^2}{a\alpha\lambda}$$

$$= -\frac{c\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a^2\alpha^3\lambda^4} \quad .$$

Then with:

$$\frac{d\lambda^i}{ds} + \Gamma^i_{\alpha,\beta}\,\lambda^\alpha \frac{dx^\beta}{ds} = K\mu^i \quad ,$$

$$K\mu^1 = \frac{c^2}{a\alpha^2\lambda^3}\,(+\;0) - a\alpha^2\lambda\,\frac{c^2}{a^2\alpha^4\lambda^4} = \frac{c^2}{a\alpha^2\lambda^3} - \frac{c^2}{a\alpha^2\lambda^3} = 0$$

$$K\mu^1 = 0 \quad .$$

$$K\mu^2 = 0 + 0 - \alpha\left(1 - \alpha^2\right)^{1/2} \cdot \frac{c^2}{a^2\alpha^4\lambda^4} \ .$$

$$K\mu^3 = -\frac{c\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a^2\alpha^3\lambda^4} + \frac{1}{a\lambda}\left(\frac{\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{\alpha\lambda}\right)\left(\frac{c}{a\alpha^2\lambda^2}\right) + 0$$

$$= -\frac{c\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a^2\alpha^3\lambda^4} + \frac{c\left(\alpha^2\lambda^2 - c^2\right)^{1/2}}{a^2\alpha^3\lambda^4}$$

$$= 0.$$

So that for the cone,

$$K\mu^i = \frac{c^2\alpha\left(1 - \alpha^2\right)^{1/2}}{a^2\alpha^4\lambda^4}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad (\lim \alpha \to 1,\ K \equiv 0).$$

Evaluating

$$|K\mu^i| = \frac{1}{a} \cdot \frac{c^2\alpha\left(1 - \alpha^2\right)^{1/2}}{\alpha^4\lambda^3} \ ,$$

NOTE: $|K| \sim \dfrac{1}{\rho^3}$.

It is easy to show that the curvature of a helix (the parametric cylindrical geodesic) is in the $\hat{i}_r$ direction only.

For the three cases (sphere, cone and cylinder) where it was shown that the surface constraint was written of the form u_i = constant (i = 1, 2 or 3; u, a curvilinear variable), the curvature $K\mu^i$ of the represented surface (assuming it existed) had a component ONLY in the $\vec{e}_i$ direction.

Whether or not this is a general property is beyond the scope of this technical note.

APPENDIX IV

40.0 SURFACES OF REVOLUTION AND LONGITUDINAL ASYMMETRY

Calculation of geodesics on surfaces of revolution and first-order perturbation due to longitudinal asymmetry (analysis in spherical coordinates) can be accomplished as follows.

Starting with the spherical surface metric:

$$\Delta s = \int \left[\left(\frac{d\rho}{d\lambda} \right)^2 + \rho^2 \left(\frac{d\phi}{d\lambda} \right)^2 + \rho^2 \sin^2 \phi \left(\frac{d\theta}{d\lambda} \right)^2 \right]^{1/2} d\lambda$$

assume the surface can be written:

$\rho = \rho(\phi)$, $[= \rho(\lambda)]$, a surface of revolution.

Then with $\phi = \tan^{-1} \lambda$, $\theta = g(\lambda)$ obtain:

$$\Delta s = \int \rho(\lambda) \cdot \left[\left(1 + \left(\frac{1}{\rho} \frac{\partial \rho}{\partial \phi} \right)^2 \right) \cdot \left(\frac{1}{1 + \lambda^2} \right)^2 + \frac{\lambda^2}{1 + \lambda^2} \; g'^2 \right]^{1/2} d\lambda$$

Let $\left(\frac{1}{\rho} \frac{\partial \rho}{\partial \phi} \right) = q(\lambda)$ then:

$$\Delta s = \int \rho(\lambda) \left[(1 + q(\lambda)) \cdot \left(\frac{1}{1 + \lambda^2} \right)^2 + \frac{\lambda^2}{1 + \lambda^2} \, g'^2 \right]^{1/2} d\lambda$$

Since the partial of the integrand with respect to g = 0, then, as done before, let

$$c = \rho(\lambda) \cdot \left[(1 + q(\lambda)) \left(\frac{1}{1 + \lambda^2} \right)^2 + \frac{\lambda^2}{1 + \lambda^2} \, g'^2 \right]^{-1/2} \frac{\lambda^2}{1 + \lambda^2} \, g'$$

where c is the first constant of integration.

Solving for g' gives:

$$g' = \pm \frac{c}{\lambda \cdot \rho(\lambda)} \cdot \left[\frac{1 + q(\lambda)}{\lambda^2 - \frac{c^2}{\rho^2(\lambda)} \left(1 + \lambda^2 \right)} \right]^{1/2}$$

and:

$$\theta(\lambda) = \theta_0 \pm c \int \frac{1}{\lambda \cdot \rho(\lambda)} \cdot \left[\frac{1 + q(\lambda)}{\lambda^2 - \frac{c^2}{\rho^2} \left(1 + \lambda^2 \right)} \right]^{1/2} d\lambda$$

θ_0, the second constant of integration.

For the sphere ρ = constant,

$q \equiv 0$,

letting $\frac{c}{\rho} = \bar{c}$

$$g' = \pm \frac{\bar{c}}{\lambda} \cdot \left[\lambda^2 \left(1 - \bar{c}^2\right) - \bar{c}^2\right]^{1/2} .$$

Letting $\dfrac{\bar{c}^2}{1 - \bar{c}^2} = \bar{\beta}^2$

$$g' = \pm \frac{1}{\lambda}\left[\frac{\lambda^2}{\beta^2} - 1\right]^{1/2}$$

$\theta = \theta_0 \pm \sec^{-1} \dfrac{\lambda}{\beta}$ as before.

For a slightly oblate planet, where e^4 terms are significant, but not e^6 + terms, proceed as follows:

$$\rho(\lambda) = b\left[1 + \frac{e^2}{2}\left(\frac{\lambda^2}{1 + \lambda^2}\right) + \frac{3e^4}{8}\left(\frac{\lambda^2}{1 + \lambda^2}\right)^2\right]$$

$$\phi(\lambda) = \tan^{-1} \lambda$$

calculate $\theta(\lambda)$ by noting

$$q(\lambda) = e^4 \cdot \frac{\dfrac{\lambda^2}{1 + \lambda^2}}{\left(1 - e^2 \cdot \dfrac{\lambda^2}{1 + \lambda^2}\right)^2}$$

so

$$\lambda\left(\frac{\lambda^2}{\beta^2} - 1\right)^{1/2} g' = \left(1 - e^2 \frac{\lambda^2}{1 + \lambda^2}\right)^{1/2} \cdot \left[1 + e^4 \frac{\dfrac{\lambda^2}{1 + \lambda^2}}{\left(1 - e^2 \cdot \dfrac{\lambda^2}{1 + \lambda^2}\right)^2}\right]^{1/2} .$$

$$\lambda\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2} g' \cong 1-\frac{e^2}{2}\cdot\frac{\lambda^2}{1+\lambda^2}+e^4\left[\frac{\lambda^2}{2\left(1+\lambda^2\right)^2}-\frac{\lambda^4}{8\left(1+\lambda^2\right)^2}\right] .$$

$$\theta=\theta_0 \pm\left\{\sec^{-1}\frac{\lambda}{\beta}-\frac{e^2}{2}\frac{\beta}{\left(1+\beta^2\right)^{1/2}}\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right.$$

$$\left.+\frac{e^4}{2}\left[\int\frac{\lambda d\lambda}{\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}\cdot\left(1+\lambda^2\right)^2}-\int\frac{\lambda^3 d\lambda}{4\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}\cdot\left(1+\lambda^2\right)^2}\right]\right\} .$$

This solves explicitly as:

$$\theta(\lambda)=\theta_0 \pm\left\{\sec^{-1}\frac{\lambda}{\beta}-\frac{e^2}{2}\frac{\beta}{\left(1+\beta^2\right)^{1/2}}\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right.$$

$$+\frac{e^4}{8}\left[\frac{5\beta}{2\left(1+\beta^2\right)^{1/2}}\cdot\left(\frac{1}{1+\lambda^2}\cdot\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}+\frac{1}{\left(1+\beta^2\right)}\right.\right.$$

$$\left.\left.\left.\cdot\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right)-\frac{\beta}{\left(1+\beta^2\right)^{1/2}}\cdot\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right]\right\} .$$

$\psi(\lambda)$ and $\Delta s(\lambda)$ can now be solved for by the usual method:

$$\psi(\lambda)=\tan^{-1}\left\{\beta\cdot\left(\frac{\lambda^2+1}{\lambda^2-\beta^2}\right)^{1/2}\cdot\left[1-\frac{e^2}{2}\cdot\frac{\lambda^2}{1+\lambda^2}-\frac{e^4}{8}\left(\frac{\lambda^2}{1+\lambda^2}\right)^2\right]\right\},$$

$$\Delta s=b\left(1-e^2\frac{\beta^2}{1+\beta^2}\right)^{1/2}\cdot\left\{\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\cdot\left(1+\frac{e^2}{2}-\frac{e^2}{4\left(1+\beta^2\right)}\right)\right.$$

$$-\frac{e^2}{4\left(1+\beta^2\right)^{1/2}}\cdot\frac{\left(\lambda^2-\beta^2\right)^{1/2}}{1+\lambda^2}+\frac{3e^4}{8}\cdot\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}$$

$$-\frac{e^4}{8}\left[\frac{1}{1+\lambda^2}\cdot\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}+\frac{1}{\left(1+\beta^2\right)}\cdot\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right]$$

$$-\frac{e^4}{64}\left[\frac{2}{\left(1+\lambda^2\right)^2}\cdot\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}+\frac{3}{\left(1+\lambda^2\right)\left(1+\beta^2\right)}\cdot\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right.$$

$$\left.\left.+\frac{3}{\left(1+\beta^2\right)^2}\cdot\tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2}\right]\right\}\Bigg|_{\lambda_0}^{\lambda_1} .$$

Longitudinal asymmetry is a very complex, highly nonlinear problem, solvable primarily via successive approximation.*

Start by assuming

$\rho = a(1 + eg)$ where $e <<1$

In the analysis assume the primary interest is in working on a small surface portion, or sectoral slice, where the above assumption is approximately true; also, assume working in the low to mid latitude region is of primary interest, thus excluding analysis in the polar region, ie, $\varepsilon^2 \sim 0$ and $\varepsilon << \frac{\lambda^2}{1 + \lambda^2}$ for all λ in the domain of interest.

On the earth such an example would be the geodesic from St Louis, Missouri (elevation ~ 1000 feet) to Denver, Colorado (elevation ~ 6000 feet).

Continuing:

$$\Delta s = \int \left[\left(\frac{d\rho}{d\lambda} \right)^2 + \rho^2 \left(\frac{d\phi}{d\lambda} \right)^2 + \rho^2 \sin^2 \phi \left(\frac{d\theta}{d\lambda} \right)^2 \right]^{1/2} d\lambda$$

$$\frac{d\rho}{d\lambda} = a \, \varepsilon \, g'$$

$$\frac{1}{\rho} \frac{d\rho}{d\lambda} \cong \varepsilon g'$$

and

$$\Delta s = a \int (1 + \varepsilon g) \left[\varepsilon^2 g'^2 + \left(\frac{1}{1 + \lambda^2} \right)^2 + \frac{\lambda^2}{1 + \lambda^2} g'^2 \right]^{1/2} d\lambda \quad .$$

The integrand can be written:

$$(1 + \varepsilon g) \left[\left(\frac{1}{1 + \lambda^2} \right)^2 + \left(\varepsilon^2 + \frac{\lambda^2}{1 + \lambda^2} \right) g'^2 \right]^{1/2} d\lambda \quad .$$

which can be further approximated as:

$$(1 + \varepsilon g) \left[\left(\frac{1}{1 + \lambda^2} \right)^2 + \frac{\lambda^2}{1 + \lambda^2} g'^2 \right]^{1/2} \quad .$$

*Reference Einstein, Law of Gravitation, pgs. 328-334; Lass, Vector and Tensor Analysis.

In this example, letting the integrand be δ, we see immediately $\frac{\partial\delta}{\partial g} \neq 0$,

Therefore we must use:

$$\frac{d}{d\lambda}\frac{\partial\delta}{\partial g'} - \frac{\partial\delta}{\partial g} = 0$$

and collect terms in ε, setting up for a successive integral approximation.

Continuing,

$$\frac{\partial\delta}{\partial g} = \varepsilon\left[\frac{1}{\left(1+\lambda^2\right)^2} + \frac{\lambda^2}{1+\lambda^2} g'^2\right]^{1/2}$$

$$\frac{\partial\delta}{\partial g'} = \frac{(1+\varepsilon g)\cdot\frac{\lambda^2}{1+\lambda^2}g'}{\left[\frac{1}{\left(1+\lambda^2\right)^2} + \frac{\lambda^2}{1+\lambda^2} g'^2\right]^{1/2}} \quad .$$

Rewriting the equation as $\frac{\partial\delta}{\partial g'} = \int \frac{\partial\delta}{\partial g} + c$

(Since as $\varepsilon \to 0$ we must obtain the unperturbed spherical solution, and letting

$$[\ \ ] = \left[\left(\frac{1}{1+\lambda^2}\right)^2 + \frac{\lambda^2}{1+\lambda^2} g'^2\right] \quad :$$

$$\frac{\lambda^2}{1+\lambda^2} g'^2 = \varepsilon\ [\ \ ]^{1/2}\cdot\left(cg + \int [\ \ ]^{1/2}\, d\lambda\right) + c\cdot[\ \ ]^{1/2} \quad .$$

The system becomes:

$$\frac{\lambda^2}{1+\lambda^2} g' - c\left[\left(\frac{1}{1+\lambda^2}\right)^2 + \frac{\lambda^2}{1+\lambda^2} g'^2\right]^{1/2} = \varepsilon\cdot\left\{[\ \ ]^{1/2}\cdot\left(cg + \int [\ \ ]^{1/2}\, d\lambda\right)\right\} \quad .$$

Start by setting $\varepsilon \equiv 0$, solving for g obtain:

$$g_0 = \theta_0 \pm \sec^{-1}\frac{\lambda}{\beta} \quad .$$

Substituting this on the right side of the equation, obtain:

$$\frac{\lambda^2}{1+\lambda^2} g' - c\ [\ \ ]^{1/2} = \varepsilon\cdot\left[\left(\frac{1}{1+\lambda^2}\right)^2 + \frac{1}{\left(\lambda^2+1\right)\left(\frac{\lambda^2}{\beta^2}-1\right)}\right]^{1/2}$$

$$\cdot \left\{ c\,\theta_0 \pm c \sec^{-1}\frac{\lambda}{\beta} + \int \left[\left(\frac{1}{1+\lambda^2} \right)^2 + \frac{1}{\left(\lambda^2+1\right)\left(\frac{\lambda^2}{\beta^2}-1\right)} \right]^{1/2} d\lambda \right\}$$

$$\frac{\lambda^2}{1+\lambda^2}\, g' - c\ [\ \ ]^{1/2} = \varepsilon(\lambda)$$

where $\varepsilon(\lambda)$ =

$$\varepsilon \cdot \frac{\lambda}{1+\lambda^2} \cdot \left(\frac{1+\beta^2}{\lambda^2-\beta^2} \right)^{1/2} \cdot \left\{ c\,\theta_0 \pm c \sec^{-1}\frac{\lambda}{\beta} + \tan^{-1}\left(\frac{\lambda^2-\beta^2}{1+\beta^2} \right)^{1/2} \right\} .$$

The next iteration for g (= g_1) can now be set up and performed numerically (note $\varepsilon(\lambda)$ has a ± term, this accounts for asymmetry in the $\hat{i}_\theta$ direction).

Now obtain:

$$\lambda^2\left(\lambda^2\left(1-c^2\right)-c^2\right) g'^2 = c^2 + 2c\,\varepsilon(\lambda) \cdot [\ \ ]^{1/2} \qquad (\varepsilon^2 \sim 0).$$

continuing the process of successive approximation:

$$\lambda^2\left(\lambda^2\left(1-c^2\right)-c^2\right) g'^2 = c^2 + 2c\,\varepsilon(\lambda) \cdot \frac{\lambda}{1+\lambda^2} \cdot \left(\frac{1+\beta^2}{\lambda^2-\beta^2} \right)^{1/2}$$

$$\lambda\left(\frac{\lambda^2}{\beta^2}-1\right) g' = \pm \left[1 + \frac{2}{c}\,\varepsilon(\lambda) \cdot \frac{\lambda}{1+\lambda^2} \cdot \left(\frac{1+\beta^2}{\lambda^2-\beta^2} \right)^{1/2} \right]^{1/2} .$$

and finally:

$$g = \theta_0 \pm \int \frac{\left(1+\varepsilon \cdot r(\lambda)\right)^{1/2}}{\lambda\left(\frac{\lambda^2}{\beta^2}-1\right)^{1/2}}\, d\lambda$$

where $r(\lambda) = \dfrac{2}{c}\,\dfrac{\lambda^2}{\left(1+\lambda^2\right)^2}\left(\dfrac{1+\beta^2}{\lambda^2-\beta^2}\right) \cdot \left\{ c\,\theta_0 \pm \sec^{-1}\dfrac{\lambda}{\beta} + \tan^{-1}\left(\dfrac{\lambda^2-\beta^2}{1+\beta^2}\right)^{1/2} \right\}$

NOTE: $\beta^2 = \dfrac{c^2}{1-c^2}$

Caution must be used when determining which combination of ±'s need to be employed, and on which side of θ_0. Once this is accomplished, $\psi(\lambda)$ and $\Delta s(\lambda)$ can be calculated in a straight forward manner.

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*principle reference used

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AFSPACECOM TECHNICAL NOTE

SURFACE NAVIGATION AND GEODESY

A PARAMETRIC APPROACH

ADDENDUM I

31 OCTOBER 1988

HEADQUARTERS AIR FORCE SPACE COMMAND

W A R N I N G

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PREFACE

This is the first addendum to AFSPACECOM Technical Note, **Surface Navigation and Geodesy, A Parametric Approach**, 1 March 1988. This addendum will address two issues not covered in the aforementioned publication.

First, a practical problem involving conic geodesy will be presented and the solution discussed.

Second, the solution for the parametric geodesic of a paraboloid (parabola of revolution) will be presented, but the analysis will involve the use of parabolic confocal coordinates.

It is hoped this addendum will further demonstrate the value of the use of parametric analysis in solving problems of surface geodesy.

CONTENTS

31 October 1988

TECHNICAL NOTE

I. AN APPLICATION OF CONIC GEODESY

In many circumstances the surfaces of certain physical features can be approximately modeled (to a very high degree) by that of a cone. A case in point would be Mount Fuji in Japan, which will be assumed to be in the shape of a perfect cone. Consider the following problem:

A group of Shinto monks are making a pilgramage from their home in Kōfu to their seaside temple at Numazu. On the route their master has selected, they will have to pass across the base of Mount Fuji, from the northern most point to the southern most point. The leader of the noble group of pilgrim monks has been charged with taking the route about or over Mount Fuji which will traverse the minimum distance. It can immediately be seen by symmetry that there will be at least two routes, one about the east face and the other about the west face. Not withstanding this temporary dilemma (which will be decided by the toss of a coin) the following elements will have to be calculated/determined:

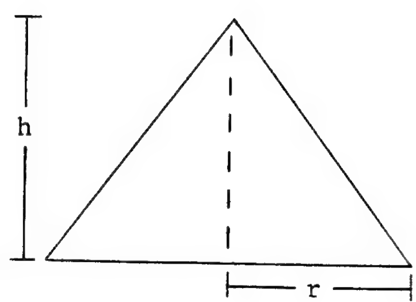
a. The minima curve.

b. The total distance traveled.

c. How high up the side of the cone will the monks need to travel.

d. The initial angle of attack.

Given, Mount Fuji has conic parameters.



So that,

$$\rho_0 = (h^2 + r^2)^{1/2}$$

$$\alpha = \frac{r}{h}$$

$$\gamma = \frac{r}{(h^2 + r^2)^{1/2}}$$

Solution. Reference paragraph IV, **Surface Navigation and Geodesy, A Parametric Approach**, and obtain the parametric conic geodesic:

$$\rho = a\lambda$$

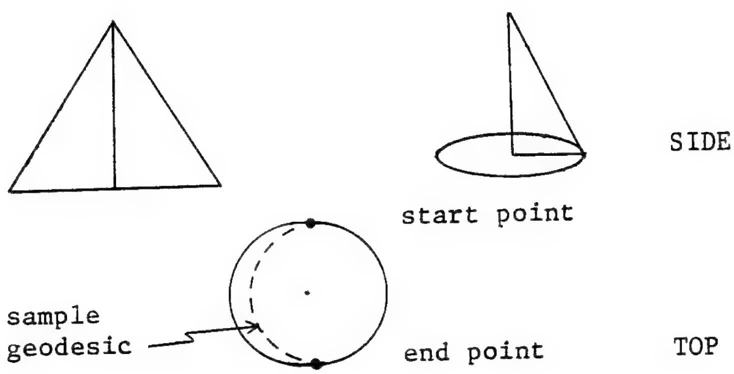
$$\phi = \tan^{-1}\alpha \quad \text{(constant)}$$

$$\theta = \theta_0 \pm \frac{1}{\gamma}\sec^{-1}\left(\frac{\gamma\lambda}{c}\right) .$$

$$\Delta s = a\left(\lambda^2 - \frac{c^2}{\gamma^2}\right)^{1/2}$$

$$\psi = \tan^{-1}\left\{\left[\left(\frac{\gamma\lambda}{c}\right)^2 - 1\right]^{-1/2}\right\} .$$

A diagram of the problem.



The selected geodesic is symmetric about the HIGH point in the route to be chosen, thus it can be concluded that ρ_{max} occurs at $\theta_0 \pm \frac{\pi}{2}$, so that

$$|\theta - \theta_0| = \frac{\pi}{2} .$$

Returning to the parametric geodesics;

$$\left(\text{from } \theta = \theta_0 \pm \frac{1}{\gamma}\sec^{-1}\left(\frac{\gamma\lambda}{c}\right)\right)$$

$$\sec\left(\gamma\,\frac{\pi}{2}\right) = \frac{\gamma\lambda_{max}}{c} .$$

$$(\text{from } \rho = a\lambda)$$

$$\lambda_{max} = \frac{\rho_0}{a}$$

so that

$$c = \frac{\gamma\rho_0}{a}\cos\left(\frac{\gamma\pi}{2}\right) \quad ,$$

this fixes the parametric curve ($\theta_0 = \frac{\pi}{2}$ for west face ascent, $\frac{3\pi}{2}$ for east face ascent.)

Now, ρ_{min} occurs at $|\theta - \theta_0| = 0$, so that

$$1 = \frac{\gamma\lambda_{min}}{c}$$

$$\frac{ac}{\gamma} = \rho_{min} = \frac{a}{\gamma}\left(\frac{\gamma}{a}\rho_0 \cos\left(\gamma\frac{\pi}{2}\right)\right) \quad .$$

From this obtain

$$\rho_{min} = \rho_0 \cos\left(\frac{\gamma\pi}{2}\right) \quad .$$

The distance, measured from ground level, traveled up the side of the mountain is

$$\rho_0 - \rho_{min} = \rho_0\left[1 - \cos\left(\frac{\gamma\pi}{2}\right)\right] \quad .$$

The total distance to be traveled from one side of the mountain to the other can be evaluated as follows:

$$\Delta s = a\left(\lambda^2 - \frac{c^2}{\gamma^2}\right)^{1/2}\Bigg|_{\lambda_0}^{\lambda_1}$$

So that, for both ascent and descent parts of the geodesic:

$$\Delta s = 2a\left(\lambda^2 - \frac{c^2}{\gamma^2}\right)^{1/2}\Bigg|_{\lambda=(\rho_{min}/a)=\lambda_{min}}^{\lambda=(\rho_0/a)=\lambda_{max}}$$

Now, since

$$\lambda_{min} = \frac{c}{\gamma} \quad ,$$

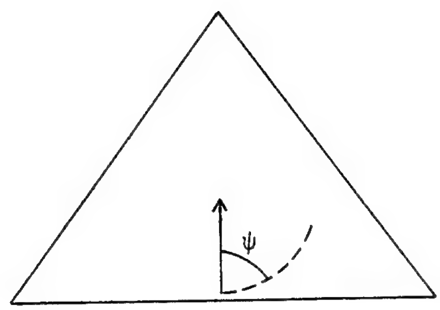
$$\lambda_{max} = \frac{\rho_0}{a} \quad ,$$

$$c = \frac{\gamma\rho_0}{a} \cos\left(\frac{\gamma\pi}{2}\right) \quad ,$$

$$\Delta s = 2a \left[\frac{\rho_0^2}{a^2} - \frac{\gamma^2\rho_0^2}{\gamma^2 a^2} \cos^2\left(\frac{\gamma\pi}{2}\right)\right]^{1/2}$$

$$\Delta s = 2\rho_0 \sin\left(\frac{\gamma\pi}{2}\right) \quad .$$

The initial angle of attack, defined as the angle as measured off from the top of the mountain, measured in the plane whose normal is parallel to the conic surface normal.



From

$$\tan\psi = \left[\left(\frac{\gamma\lambda}{c}\right)^2 - 1\right]^{-1/2}$$

$$\frac{\gamma\lambda_{max}}{c} = \sec\left(\frac{\gamma\pi}{2}\right) \quad .$$

$$\tan\psi = \cot\left(\frac{\gamma\pi}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{\gamma\pi}{2}\right)$$

$$\psi = \frac{\pi}{2}(1 - \gamma) \quad .$$

Summarizing the solutions:

Total distance traveled = $2\rho_0 \sin\left(\frac{\gamma\pi}{2}\right)$.

Distance of ascent up the mountain side = $\rho_0 \left[1 - \cos\left(\frac{\gamma\pi}{2}\right)\right]$.

Initial angle of attack = $\frac{\pi}{2}(1 - \gamma)$.

II. SOLUTION OF THE PARAMETRIC GEODESIC FOR A PARABOLOID (SURFACE OF A PARABOLA OF REVOLUTION)

This section will be brief, outlining only the most necessary steps required to get to our objective, that being, defining a surface constraint and develop the five equation parametric solution for the geodesic curves (the necessary surface navigation relations).

For simplicity sake, the surface defined in cylindrical coordinates is:

$$r^2 = 2a\left(z + \frac{a}{2}\right) ,$$

where a is an arbitrary parameter.

Parabolic confocal coordinates can be described as the set (ξ, η, θ) such that θ is the same longitudinal variable as described in cylindrical or spherical coordinates, then using ρ from spherical coordinates and Cartesian z,

$$\xi = \rho - z$$

$$\eta = \rho + z$$

and the surface constraint equation can be transformed to:

$$r^2 = a(2z + a)$$

$$z^2 + r^2 = z^2 + 2az + a^2,$$

$$z^2 + r^2 = \rho^2$$

$$\rho^2 = (z + a)^2$$

$$\rho = z + a$$

$$\rho - z = a = \xi$$

hence, the surface constraint is equivalent to setting one of the curvilinear variables to a constant, namely $\xi = a$.

Since the Jacobian matrix is formed to show curvilinearity and obtain the scaling factors, it is convenient to work with spherical coordinates as an intermedium:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{pmatrix} = \begin{pmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ z \end{pmatrix} .$$

From

$\rho - z = \xi$

$\rho + z = \eta$

$2z = \eta - \xi$

so that

$z = \frac{1}{2}(\eta - \xi)$.

Now

$\rho = \frac{1}{2}(\eta + \xi)$

so that

$$\cos\phi = \frac{z}{\rho} = \frac{\eta - \xi}{\eta + \xi} \quad .$$

$$\sin\phi = \left(1 - \left(\frac{\eta - \xi}{\eta + \xi}\right)^2\right)^{1/2}$$

$$= \frac{1}{\eta + \xi}\left(\eta^2 + 2\eta\xi + \xi^2 - \eta^2 + 2\eta\xi + \xi^2\right)^{1/2}$$

$$= \frac{2(\eta\xi)^{1/2}}{\eta + \xi}$$

$$x = \rho \sin\phi \cos\theta$$

$$= \tfrac{1}{2}(\eta + \xi)2\,\frac{(\eta\xi)^{1/2}}{(\eta + \xi)}\cos\theta$$

$$= (\eta\xi)^{1/2}\cos\theta$$

A similar relation exists for y, so that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (\eta\xi)^{1/2} \cos\theta \\ (\eta\xi)^{1/2} \sin\theta \\ \frac{1}{2}(\eta - \xi) \end{pmatrix} \quad .$$

The Jacobian matrix can be formed by taking the appropriate partials:

$$J = \begin{Bmatrix} \frac{1}{2}\left(\frac{\xi}{\eta}\right)^{1/2}\cos\theta & \frac{1}{2}\left(\frac{\eta}{\xi}\right)^{1/2}\cos\theta & -(\eta\xi)^{1/2}\sin\theta \\ \frac{1}{2}\left(\frac{\xi}{\eta}\right)^{1/2}\sin\theta & \frac{1}{2}\left(\frac{\eta}{\xi}\right)^{1/2}\sin\theta & (\eta\xi)^{1/2}\cos\theta \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{Bmatrix}$$

It is now a simple matter to show that, letting I_i represent the vector of the ith column in J,

$$I_i \cdot I_j = \left.h_i\right] \delta_{ij} \left[h_j\right.$$

where

$$\delta_{ij} = \begin{pmatrix} 0 & i \neq j \\ 1 & i = j \end{pmatrix} \quad .$$

Thus, the coordinates are truly curvilinear and the scaling factors may be obtained as:

$$h_1 = \tfrac{1}{2}\left(1 + \left(\frac{\xi}{\eta}\right)\right)^{1/2}$$

$$h_2 = \tfrac{1}{2}\left(1 + \left(\frac{\eta}{\xi}\right)\right)^{1/2}$$

$$h_3 = (\eta\xi)^{1/2}$$

The distance equation becomes:

$$(ds)^2 = \frac{1}{4\eta}(\eta + \xi)(d\eta)^2 + \frac{1}{4\xi}(\eta + \xi)(d\xi)^2 + \eta\xi(d\theta)^2 \quad .$$

Rewriting parametrically, with $\xi = a$

$$s = \int \left[\frac{1}{4\eta}(\eta + a)\left(\frac{d\eta}{d\lambda}\right)^2 + \eta a\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda \quad .$$

There are a variety of ways which can be chosen to solve this equation, for example

$$s = \int \left[\frac{1}{4\eta}(\eta + a)\left(\frac{d\eta}{d\lambda}\right)^2 + \eta a\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda \quad .$$

Let

$$\eta = af$$

$$\theta = g$$

$$s = a\int \left[\frac{1}{4f}(f + 1)f'^2 + fg'^2\right]^{1/2} d\lambda$$

and since the integrand is not dependent on g explicitly, then letting δ represent the integrand,

set

$$\frac{\partial \delta}{\partial g'} = c \quad .$$

$$\frac{fg'}{\left[\frac{1}{4f}(f + 1)f'^2 + fg'^2\right]^{1/2}} = c$$

$$f^2g'^2 = c^2\left[\frac{1}{4f}(f + 1)f'^2 + fg'^2\right]$$

$$\left[f^2 - fc^2\right] g'^2 = \frac{c^2}{4f}(f + 1)f'^2$$

$$g' = \frac{cf'}{2f}\left[\frac{f + 1}{f - c^2}\right]^{1/2}$$

Let

$$\frac{f + 1}{f - c^2} = \lambda^2$$

$$f + 1 = \lambda^2 f - \lambda^2 c$$

$$f = \frac{\lambda^2 c^2 + 1}{\lambda^2 - 1} \quad ,$$

$$f' = \frac{1}{(\lambda^2 - 1)^2}\left[(\lambda^2 - 1)2\lambda c^2 - 2\lambda(\lambda^2 c^2 + 1)\right]$$

$$= \frac{2\lambda}{(\lambda^2 - 1)^2}\left[\lambda^2 c^2 - c^2 - \lambda^2 c^2 - 1\right]$$

$$= \frac{-2\lambda(c^2 + 1)}{(\lambda^2 - 1)^2} \quad .$$

Continue with

$$g = \theta_0 \pm \frac{c}{2}\int \frac{f'}{f}\, \lambda d\lambda$$

$$= \theta_0 \pm \frac{c}{2}\int \frac{-2\lambda(c^2 + 1)}{(\lambda^2 - 1)^2} \cdot \frac{(\lambda^2 - 1)}{\lambda^2 c^2 + 1}\, \lambda d\lambda$$

$$= \theta_0 \pm c(c^2 + 1)\int \frac{\lambda^2 d\lambda}{(\lambda^2 - 1)(\lambda^2 c^2 + 1)}$$

now

$$\frac{1}{(x-1)(ax+1)} = \frac{1}{1+a}\left[\frac{1}{x-1} - \frac{a}{ax+1}\right]$$

so that letting

$a = c^2$

$x = \lambda^2$

$$g = \theta_0 \pm c \int \lambda^2 d\lambda \left(\frac{1}{\lambda^2 - 1}\right) - \left(\frac{c^2}{\lambda^2 c^2 + 1}\right) \quad .$$

Solving the integral:

$$\int \frac{\lambda^2}{\lambda^2 - 1} - \frac{c^2\lambda^2}{c^2\lambda^2 + 1}\, d\lambda$$

$$\int \frac{\lambda^2}{\lambda^2 - 1}\, d\lambda = - \int \frac{\lambda^2}{1 - \lambda^2}\, d\lambda$$

Let

$\lambda = \tanh u$

$d\lambda = \operatorname{sech}^2 u d\lambda$

$$\begin{aligned}\int \frac{\lambda^2}{\lambda^2 - 1}\, d\lambda &= - \int \tanh^2 u du \\ &= - \int 1 - \operatorname{sech}^2 u du \qquad (\operatorname{sech}^2 u - 1) \\ &= - (u - \tanh u) \\ &= (\lambda - \tanh^{-1} \lambda)\end{aligned}$$

$$- \left[\int \frac{c^2\lambda^2}{c^2\lambda^2 + 1}\, d\lambda \right]$$

Let

$c\lambda = \tan \theta$

$c^2\lambda^2 = \tan^2 \theta$

$d\lambda = \frac{1}{c} \sec^2 \theta d\theta$

obtain:

$$-\left[\frac{1}{c}\int \tan^2 \theta d\theta\right.$$

$$-\left[\frac{1}{c}\int \sec^2 \theta - 1\, d\theta\right.$$

$$-\left[\frac{1}{c}(\tan\theta - \theta)\right] = -\left[\frac{1}{c}\left| c\lambda - \tan^{-1} c\lambda \right|\right]$$

$$g = \theta_0 \mp c\left[(\lambda - \tanh^{-1}\lambda - \lambda - \frac{1}{c}\tan^{-1} c\lambda\right] .$$

$$\theta = \theta \pm \left[c \tanh^{-1}\lambda - \tan^{-1} c\lambda\right] .$$

Recapping the first two coordinate variables:

$$\xi = a$$

$$\eta = a\left[\frac{\lambda^2 c^2 + 1}{\lambda^2 - 1}\right]$$

In this example, one constant of integration, c, appears in two coordinate variables, however, the curve is still symmetric about an arbitrary θ_0, as one would expect from a surface of revolution.

Continuing for ψ,

$$\vec{D} = h_1 \frac{d\eta}{d\lambda}\hat{e}_1 + h_3 \frac{d\theta}{d\lambda}\hat{e}_3$$

$$\tan\psi = \frac{\vec{D}\cdot\hat{e}_3}{\vec{D}\cdot\hat{e}_1}$$

which resolves as:

$$\eta = a\left[\frac{\lambda^2 c^2 + 1}{\lambda^2 - 1}\right]$$

$$\frac{d\eta}{d\lambda} = \frac{-2a\lambda(c^2 + 1)}{(\lambda^2 - 1)^2}$$

$$\frac{d\theta}{d\lambda} = \frac{c(c^2 + 1)\lambda^2}{(\lambda^2 - 1)(\lambda^2 c^2 + 1)}$$

$$h_3 = (\eta\xi)^{1/2} = a\left(\frac{\lambda^2 c^2 + 1}{\lambda^2 - 1}\right)^{1/2}$$

$$h_1 = \tfrac{1}{2}(\eta + \xi)^{1/2} = \frac{(a)^{1/2}}{2}\left(1 + \frac{\lambda^2 c^2 + 1}{\lambda^2 - 1}\right)^{1/2}$$

and obtain:

$$\psi = \tan^{-1}\left[\frac{c}{(1 + c^2)^{1/2}} \cdot (\lambda^2 - 1)^{1/2}\right] \quad .$$

Finally, for distance transversed:

$$\Delta s = \int \left[\frac{1}{4\eta}(\eta + a)\left(\frac{d\eta}{d\lambda}\right)^2 + \eta a\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda$$

$$\Delta s = a\int\left[\frac{\lambda^2 - 1}{4(\lambda^2 c^2 + 1)} \cdot \left(\frac{\lambda^2 c^2 + 1}{\lambda^2 - 1} + 1\right) \cdot \frac{4\lambda^2 (c^2 + 1)^2}{(\lambda^2 - 1)^4}\right.$$

$$\left. + \frac{\lambda^2 c^2 + 1}{\lambda^2 - 1}\left(\frac{c^2 (c^2 + 1)^2 \lambda^4}{(\lambda^2 - 1)^2 (\lambda^2 c^2 + 1)^2}\right)\right]^{1/2} d\lambda$$

$$\Delta s = a\int\left(\frac{\lambda^4 (c^2 + 1)^2}{(\lambda^2 - 1)^2}\right)^{1/2} d\lambda$$

$$\Delta s = a(c^2 + 1)\int\frac{\lambda^2 d\lambda}{(\lambda^2 - 1)^2} = a(c^2 + 1)\int\frac{\lambda^2 d\lambda^2}{(1 - \lambda^2)^2}$$

$$\Delta s = \frac{a(c^2 + 1)}{4}\left[\sinh 2\tanh^{-1}\lambda - 2\tanh^{-1}\lambda\right]\Bigg|_{\lambda} \quad .$$

λ w/appropriate change of limits.

AFSPACECOM TECHNICAL NOTE

SURFACE NAVIGATION AND GEODESY

A PARAMETRIC APPROACH

ADDENDUM II

1 MAY 1989

HEADQUARTERS AIR FORCE SPACE COMMAND

W A R N I N G

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PREFACE

This is the second addendum to AFSPACECOM Technical Note, **Surface Navigation and Geodesy, A Parametric Approach**, 1 March 1988. This addendum will address two classes of problems not previously covered.

First, a dicussion of the surface overtake problem is presented. This is a classical naval problem, whereby a submarine is sighted by an intercept craft; the submarine submerges and, at an assumed constant velocity, races away at some random angle θ (measured with respect to the initial line of sight of the intercept craft). Assuming the velocity of the intercept craft to be greater than that of the submarine, what is desired is the curve the intercept craft must take to eventually overtake and catch the submarine (this is a well known curve). The presented problem, solved on a local planar surface, also addresses certain quantities of operations research interest, such as mean distance to intercept, variance, mean time to intercept, etc. The overtake curve for a sperical surface is then solved for and presented.

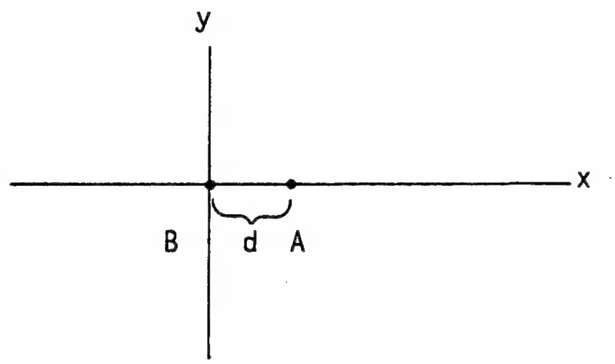
Second, the problem of minimum distance navigation on composite surfaces will be addressed; specifically, navigation on a sphere which contains a spherical dent on its surface (an approximation of the surface of the martian moon, Phobus).

CONTENTS

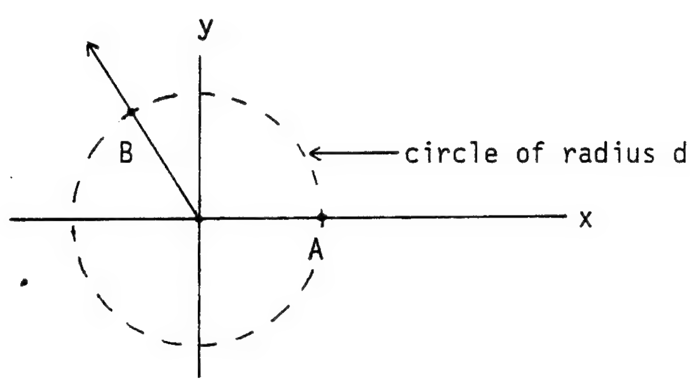
I. SURFACE OVERTAKE PROBLEM

This is a special application of the general curves of pursuit problem (for example, reference Differential Equations with Applications[1]). In this case, for a given surface, an interceptee vehicle is sighted by the interceptor. It is assumed the interceptee takes a minimum distance route away from where he was sighted, at some random heading angle.

a. Solution on the plane.



We set up the coordinate system as follows, at time $t = -\left(\frac{d}{v_0}\right)$, the interceptor is at point A, the interceptee at the origin (point B), at a distance d from the interceptor. The interceptee takes a linear route away from the origin at velocity v_0, so that at time t = 0 the interceptee is at a distance d from the origin but at an unknown (heading) location.



The equation of motion for the interceptee is (in polar coordinates)

$\theta = \bar{\theta}_0$ (unknown)

$r = r_0 + v_0 t$ (where $r_0 = d$).

[1]Ritger and Rose, **Differential Equations with Applications**, p. 94.

Assuming the interceptor travels at constant velocity v_1 ($v_1 > v_0$), the polar equation of motion for the inteceptor becomes

$$\dot{s}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \qquad (\dot{s}^2 = v_1^2)$$

and for eventual overtake and capture, the interceptor's r(t) component must match exactly to that of the interceptee, thus

$$v_1^2 = v_0^2 + (r_0 + v_0 t)^2\ \dot{\theta}^2$$

$$\frac{1}{r_0} \cdot \frac{\left(v_1^2 - v_0^2\right)^{1/2}}{1 + \frac{v_0}{r_0} t} = \dot{\theta}$$

Letting,

$$u = 1 + \frac{v_0}{r_0} t \quad ,$$

$$\frac{1}{\alpha} = \left(\frac{v_1^2}{v_0^2} - 1\right)^{1/2}$$

obtain

$$\frac{1}{\alpha}\frac{du}{u} = d\theta$$

and solving, this becomes:

$$u = u_0 e^{\alpha|\Delta\theta|} \ .$$

Substituting back for r

$$r = r_0 e^{\alpha|\Delta\theta|} \quad ,$$

or parametrically, the solution curve for the interceptor is

$$r = r_0 + v_0 t$$

$$\theta = \frac{1}{\alpha} \ln \left| 1 + \frac{v_0 t}{r_0} \right| \qquad (\text{choosing } \theta_0 \equiv 0).$$

b. Statistical results of the planar solution.

With the interceptee heading angle ($\bar{\theta}_0$) defined as a random variable, it is a reasonable question to ask what are the mission statistics for the average mission (eg, average distance and time to intercept, and variance values). Borrowing heavily from Parzen[2] we look at

$$r = r_0 e^{\alpha\theta}$$

for the solution curve, then letting θ be a random variable

$$d\theta = \frac{1}{\alpha}\frac{dr}{r}$$

where $r_0 \le r \le r_0 e^{2\pi\alpha}$ assuming an eventual overtake no later than at $\theta = 2\pi$.

Normalization for the distribution on r becomes

$$1 = c\int_{r^0}^{r_0 e^{2\pi\alpha}} \frac{dr}{r}, \qquad c = \frac{1}{2\pi\alpha}$$

so the density function $\rho(r)$ is

$$\rho(r) = \begin{cases} \dfrac{1}{2\pi\alpha}\cdot\dfrac{1}{r} & r_0 \le r \le r_0 e^{2\pi\alpha} \\ 0 & \text{otherwise} \end{cases}$$

Solving for the average r

$$\langle r\rangle = \int r\rho(r)\,dr = \frac{r_0}{2\pi\alpha}\left(e^{2\pi\alpha} - 1\right)$$

Solving for the variance of r; first

$$\langle r^2\rangle = \int r^2\rho(r)\,dr = \frac{r_0}{2\pi\alpha}\left(e^{4\pi\alpha} - 1\right)$$

and the variance

$$\delta^2 = \langle r^2\rangle - \langle r\rangle^2$$

becomes

[2]Parzen, **Modern Probability Theory and Its Applications**, pp. 308-316.

$$\delta^2 = \frac{r_0^2}{4\pi\alpha}\left[e^{4\pi\alpha}\left(1 - \frac{1}{\pi\alpha}\right) + \frac{2}{\pi\alpha}\, e^{2\pi\alpha} - \left(1 + \frac{1}{\pi\alpha}\right)\right] \quad .$$

With the average radial distance, the time to intercept becomes:

$$\langle r\rangle = r_0 + v_0 t_e \quad , \; t_e = \frac{1}{v_0}\left[\langle r\rangle - r_0\right]$$

where t_e is the average intercept time (v_0, r_0 and $\langle r\rangle$ known).

With t_e, the average total distance traveled per mission becomes:

$$\bar{s} = v_1 \cdot t_e$$

$$\bar{s} = \frac{r_0}{2\pi\alpha}\left(\frac{v_1}{v_0}\right)\left[e^{2\pi\alpha} - (1 + 2\pi\alpha)\right] \quad .$$

If $v_1 \gg v_0$ (hence $1/\alpha \gg 1$),

$$\bar{s} \approx \pi r_0 \left[1 + \frac{1}{2}\left(\frac{v_0}{v_1}\right)^2\right] \quad .$$

[Note, ordinarily the density function would be a function of s, the total distance traveled, not r; however, for $r = r_0 e^{\alpha\theta}$ r and s are linearly related:

$$\left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2 \quad .$$

Now,

$$r = r_0 e^{\alpha\theta},$$

$$\frac{dr}{d\theta} = \alpha r_0 e^{\alpha\theta} = \alpha r$$

and

$$r^2\left(\frac{d\theta}{dr}\right)^2 = \frac{1}{\alpha^2}$$

so

$$ds = dr\left(1 + \frac{1}{\alpha^2}\right)^{1/2} .$$

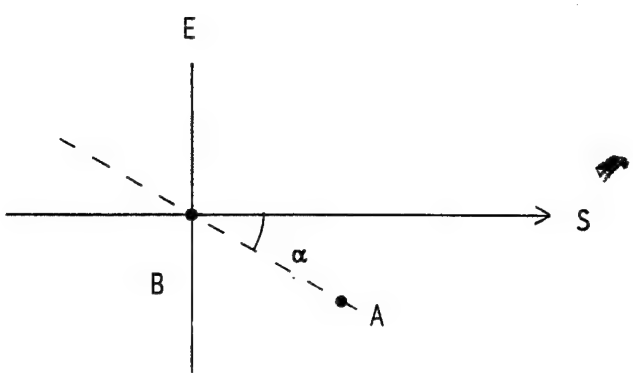
With s and r linearly related, the developed statistics do not change.]

c. Solution on the sphere.

In this case it is assumed the interceptee takes a great circle route at some unknown heading away from the origin (defined here as in the last example, the point where the interceptor saw the interceptee). Let's assume the origin is at some known latitude and longitude $\{L_a, L_o\}$. We rotate the sphere so that the origin is at the north pole of the primed system (we do this using a topocentric rotation matrix[3]).

$$M_0 = \begin{pmatrix} \sin L_a \cos L_0 & \sin L_a \sin L_0 & -\cos L_a \\ -\sin L_0 & \cos L_0 & 0 \\ \cos L_a \cos L_0 & \cos L_a \sin L_0 & \sin L_a \end{pmatrix}$$

This rotation produces a figure similar to the one shown below

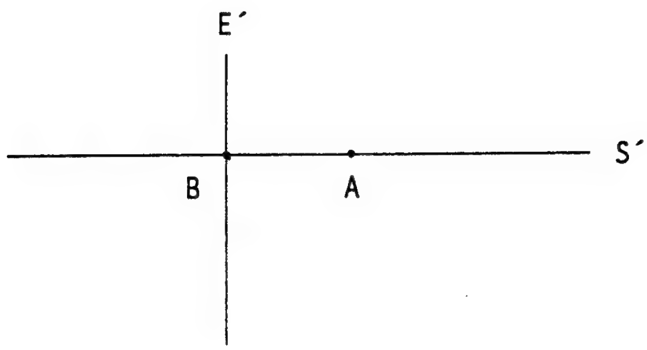


Using one more rotation about the primed polar axis

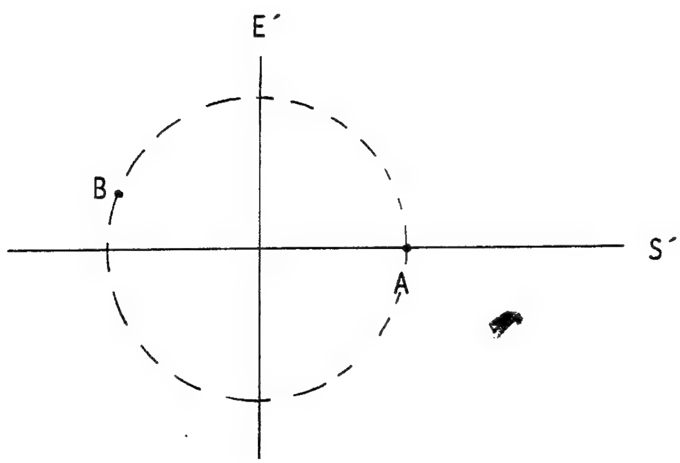
$$M_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[3]Mueller, Bate and White, **Fundamentals of Astrodynamics**, p. 79.

this aligns the interceptor A along the new ($\theta = 0$) axis as is shown below



where (for a sphere of radius a) the distance between B and A is $d = a \cdot \phi_0$. If B is at the origin at time $t = -\left(\frac{a \cdot \phi}{v_0}\right)$ then at time $t = 0$, B will be at some random heading, a distance $d = a \cdot \phi_0$ away from the origin:



In this coordinate system, a great circle is represented by $\theta = \bar{\theta}_0$ [consider a vehicle at the real north pole, any great circle route would be due south, along a line of constant longitude]. Thus in this coordinate system, the interceptee's equation of motion is

$$\theta = \bar{\theta}$$

$$\phi = \phi_0 + \frac{v_0}{a} t$$

Again assuming the interceptor travels at constant velocity v_1 ($v_1 > v_0$), the equation of motion becomes

$$\dot{s}^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \rho^2 \sin^2\phi \ \dot{\theta} \qquad \text{(and since } \rho = a)$$

$$v_1^2 = a^2\dot{\phi}^2 + a^2 \sin^2\phi \ \dot{\theta}^2$$

In analogy to the planar case, the ϕ components of the two curves must match, hence:

$$v_1^2 = v_0^2 + a^2 \sin^2 \left(\phi_0 + \frac{v_0}{a} t\right) \dot{\theta}^2$$

$$\left(\frac{v_1^2 - v_0^2}{a^2}\right)^{1/2} = \sin\left(\phi_0 + \frac{v_0}{a} t\right) \dot{\theta}$$

$$d\theta = dt \cdot \csc\left(\phi_0 + \frac{v_0}{a} t\right) \cdot \left(\frac{v_1^2 - v_0^2}{a^2}\right)^{1/2}$$

Letting

$$u = \phi_0 + \frac{v_0}{a} t$$

and

$$\frac{1}{\alpha} = \left(\frac{v_1^2}{v_0^2} - 1\right)^{1/2}$$

$$\alpha \ d\theta = \csc u \ du$$

$$\alpha\theta = c - \ln \left| \frac{1 + \cos u}{\sin u} \right|$$

$$\ln \left| \frac{\sin u}{1 + \cos u} \right| = \alpha\theta - c$$

$$\frac{\sin u}{1 + \cos u} = Ae^{\alpha\theta} \quad ,$$

but

$$\frac{\sin u}{1 + \cos u} = \tan \frac{u}{2} \quad ,$$

and setting $\theta = 0$ at $t = 0$ obtain:

$$\phi = 2 \tan^{-1} \left[\tan \frac{\phi_0}{2} e^{\alpha\theta}\right]$$

or, parametrically

$$\phi(t) = \phi_0 + \frac{v_0}{a} t$$

$$\theta(t) = \frac{1}{\alpha} \left[\ln \tan \left| \frac{\phi_0 + \frac{v_0}{\alpha} t}{2} \right| - \ln \tan \left| \frac{\phi_0}{2} \right| \right] .$$

with (ϕ, θ) solved for in the rotated (or primed) system, the solution in the original (unrotated) system can be easily generated, this by creating the primed vector

$$\vec{p}' = \begin{Bmatrix} \sin \phi \quad \cos \theta \\ \sin \phi \quad \sin \theta \\ \cos \phi \end{Bmatrix} ,$$

then finding

$$\vec{p} = (M_1 M_0)^{-1} \vec{p}' .$$

Interceptor latitude and longitude data is available directly from

$$\vec{p} = \begin{Bmatrix} \cos L_a \quad \cos L_0 \\ \cos L_a \quad \sin L_0 \\ \sin L_a \end{Bmatrix} .$$

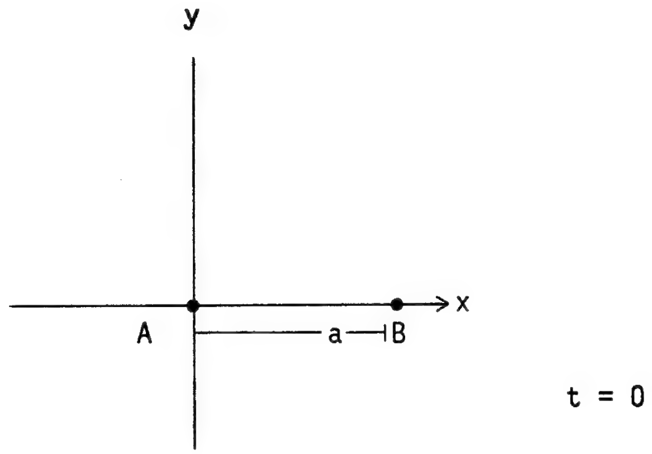
Mission statistics (averages and variances for a randomly headed interceptee) can be calculated, albeit numerically, as before.

d. Direct intercept problem.

This is the standard curve of pursuit problem, whereby the interceptor sees the interceptee, and pursues the interceptee along a curve such that the interceptor's heading is constrained to be simultaneous with the line of sight to the interceptee.

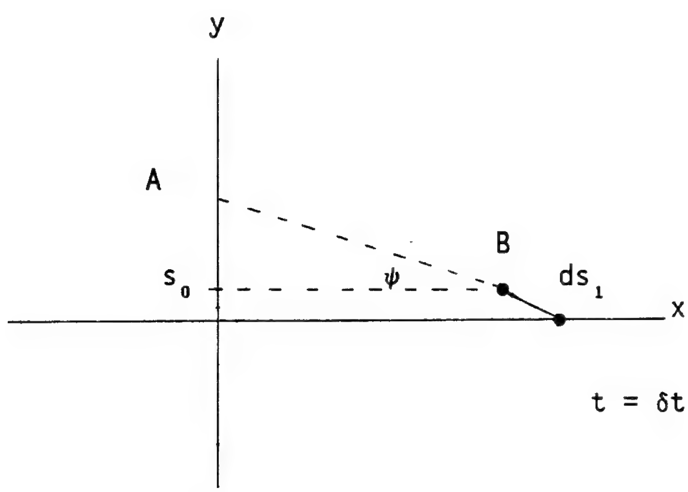
(1) Solution on the plane.

Construct the coordinate system as follows:



where A is the interceptee, initially at the origin, and B is the interceptor initially located a distance a from the origin along the x axis. Starting the clock at time t = 0, then at time $t = \delta t$ we assume the interceptee has traveled some distance δs_0 along the y axis at velocity v_0 and, similarly, the interceptor has traveled some distance δs_1, at velocity v_1.

Refering to the figure below:



In time δt the interceptor has moved

$$\frac{\delta s_1}{v_1} = \frac{1}{v_1}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \quad .$$

And the interceptee has moved $\dfrac{\delta(s_0)}{v_0}$ in the same time.

Now, distance s_0, measured from point B on the curve $y = f(x)$, is equal to

$y - x \tan \psi = y - x \frac{dy}{dx}$, and equating the interceptor to interceptee movement

in time δt:

$$\frac{d\left(y - x\frac{dy}{dx}\right)}{v_0} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx}{v_1}$$

Letting $\alpha = \frac{v_0}{v_1}$, and assuming $\alpha < 1$ $(v_0 < v_1)$

$$\frac{d}{dx}\left(y - x\frac{dy}{dx}\right) = \alpha\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$- x\frac{d^2y}{dx^2} = \alpha\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} .$$

with initial conditions:

$\frac{dy}{dx}(x = a) = 0$,

$y(x = a) = 0$,

it is relatively easy to show the solution curve is:

$$y = a\left(\frac{\alpha}{1 - \alpha^2}\right) + \frac{a}{2}\left[\frac{1}{1 + \alpha}\left(\frac{x}{a}\right)^{1+\alpha} - \frac{1}{1 - \alpha}\left(\frac{x}{a}\right)^{1-\alpha}\right]^*$$

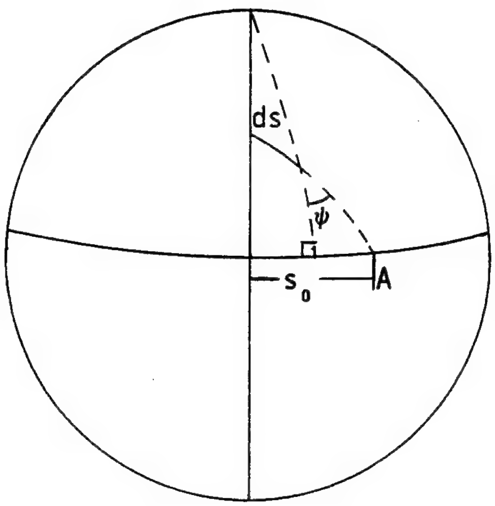
(2) Approximate solution on the sphere. As a first order approximation, we will make the following assumptions:

(a) The interceptee is located on the equator and traveling due east along the great circle route [there is no loss in generality to this assumption, providing the interceptee is confined to travel on a great circle route. This being assumed true, then a judicious choice of coordinates (ie, rotating the sphere) will put the interceptee on the equator as desired].

(b) The initial position of the interceptee is analogous to that of the planar case, in that the initial line of sight of the interceptor is perpendicular to the line of travel of the interceptee.

*Ritger and Rose, **Differential Equations with Applications**, p. 512.

(c) The initial displacement of the interceptor from the interceptee is small so that certain trigonometric approximations can be made [eg, spherical position of the interceptor {in ϕ, θ coordinates} is such that $\sin \phi \approx 1$, this being true when the interceptor position is close to the equator; eg, $\phi = 90° \pm 20°$]. Continuing as before, at some time δt, the placement of interceptor and interceptee will be as follows:



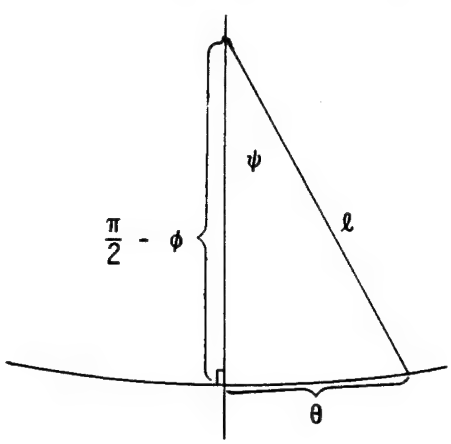
Now, for point B

$$\delta t = \frac{1}{v_1}\left[1 + \sin^2 \phi \left(\frac{d\theta}{d\phi}\right)^2\right]^{1/2} d\phi \cdot a$$

where a is the spherical radius, and measuring s_0 from point B

$$\frac{s_0}{a} = \theta + \tan^{-1}\left[\cos \phi \sin \phi \left(\frac{d\theta}{d\phi}\right)\right]$$

The second term is obtained from spherical trigonometry



$$\cos \ell = \sin \phi \cos \theta$$
$$\sin \theta = \sin \ell \sin \psi$$
$$\cos \theta = \cos \ell \sin \phi + \sin \ell \cos \phi \cos \psi$$

$\tan\theta = \cos\phi \tan\psi$

and from Carton-Wippern[4], where:

$$\tan\psi = \sin\phi \frac{d\theta}{d\phi} \quad .$$

Equating both sides at time δt:

$$\frac{1}{V_1}\left[1 + \sin^2\phi\left(\frac{d\theta}{d\phi}\right)^2\right]^{1/2} d\phi = \frac{1}{V_0}\, d\left[\theta + \tan^{-1}\left(\sin\phi\cos\phi\frac{d\theta}{d\phi}\right)\right] \quad .$$

Now, using the assumption from Id(2)(c) and by assuming that the term $\cos\phi\left(\frac{d\theta}{d\phi}\right)$ remains small so as to approximate $\tan^{-1} u \cong u$, the equation of the pursuit curve becomes:

$$\frac{V_0}{V_1}\left[1 + \left(\frac{d\theta}{d\phi}\right)^2\right]^{1/2} = \frac{d}{d\phi}\left[\theta + \cos\phi\left(\frac{d\theta}{d\phi}\right)\right] \quad .$$

Continuing on the right side with:

$$\frac{d\theta}{d\phi} + \cos\phi\frac{d^2\theta}{d\phi^2} - \sin\phi\frac{d\theta}{d\phi} \cong \cos\phi\frac{d^2\theta}{d\phi^2}$$

and letting

$$\alpha = \frac{V_0}{V_1}\ (\alpha < 1) \quad ,$$

$$p = \frac{d\theta}{d\phi} \quad ,$$

$$\alpha[1 + p^2]^{1/2} = \cos\phi\frac{dp}{d\phi}$$

$$\alpha\sec\phi\, d\phi = \frac{dp}{[1 + p^2]^{1/2}}$$

$$\alpha \ln|\sec\phi + \tan\phi| = \ln|p + (p^2 + 1)^{1/2}| + c$$

Letting $p = 0$ at $\phi = \phi_0$, then

[4]Carlton-Wippern, **Surface Navigation and Geodesy, A Parametric Approach**, pp. 38-43.

$$p + (p^2 + 1)^{1/2} = \left[\frac{\tan(x)}{\tan(x_0)}\right]^{\alpha}$$

where

$$x = \frac{\phi}{2} + \frac{\pi}{4} \quad ,$$

$$x_0 = \frac{\phi_0}{2} + \frac{\pi}{4} \quad .$$

Solving for p, the differential equation for θ is

$$\frac{d\theta}{d\phi} = \frac{1}{2}\left[(\tan x_0)^{-\alpha} \cdot (\tan x)^{\alpha} - (\tan x_0)^{\alpha} \cdot (\tan x)^{-\alpha}\right] \quad .$$

Thus, the integral equation for θ as a function of ϕ, assuming convergence, becomes:

$$\theta = \theta_0 + \frac{1}{2}\left|\int_{\phi_0}^{\phi} \left[(\tan x_0)^{-\alpha} \cdot (\tan x)^{\alpha} - (\tan x_0)^{\alpha} \cdot (\tan x)^{-\alpha}\right] d\phi\right|$$

where, for the interceptor, $\theta = \theta_0$, $\phi = \phi_0$ at time $t = 0$.

To show that the integral will, in fact, converge it is necessary to change variables, $\phi = \frac{\pi}{2} - L$, and use L, the conventional latitude variable.

Then

$$\tan^{\alpha}\left(\frac{\phi}{2} + \frac{\pi}{4}\right) = \tan^{-\alpha}\left(\frac{L}{2}\right)$$

and the integral becomes

$$\theta = \theta_0 + \frac{1}{2}\left|\int_{L_0}^{L} \tan^{-\alpha}\left(\frac{L_0}{2}\right)\tan^{\alpha}\left(\frac{u}{2}\right) - \tan^{\alpha}\left(\frac{L_0}{2}\right)\tan^{-\alpha}\left(\frac{u}{2}\right) du\right| \quad ,$$

where as the interceptor approaches the interceptee, L (or U) $\rightarrow$ 0. In this limit, it is easy to show that the above integral (in fact, the entire spherical analysis) converges to the planar case, on the local topocentric plane.

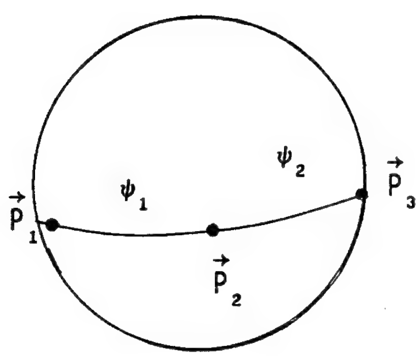
II OVERVIEW OF MINIMUM DISTANCE NAVIGATION ON COMPOSITE SURFACES

In this section, we will confine our attention to solving for the curve of minimum distance between two points when the surface on which the solution is sought can be described as two simple surfaces (eg, a sphere cut in half, so that the surface is a hemisphere bounded on top by a plane). We let S_1 be the first simple surface with geodesic ψ_1, S_2 be the second simple surface with geodesic ψ_2 and let $c = c(u, \lambda)$ be the locus of points which make up the boundary between the two surfaces. If the geodesic between two points traverse the boundary c at say (u_0, λ_0), then $\psi_1(u_0, \lambda_0) = \psi_2(u_0, \lambda_0)$ by continuity. Futhermore, if the boundary is smooth and continuous,

$$\vec{T}_1 = \frac{\partial \psi_1}{\partial s} \cdot \Big| (u_0, \lambda_0) = \vec{T}_2 = \frac{\partial \psi_2}{\partial s} \Big| (u_0, \lambda_0) \quad ,$$

the tangent must also be continuous. While these conditions are necessary, they are not sufficient. The problem is a nonlinear one, requiring judicious evaluation.

<u>Example</u>. Let $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}_3$ be three points on the equator of a sphere equally spaced (ie, $\vec{P}_1$ at 0° longitude, $\vec{P}_2$ at 120° longitude and $\vec{P}_3$ at 240° longitude). The geodesic (ψ_1) between $\vec{P}_1$ and $\vec{P}_2$ lies along the equator, from $\vec{P}_1$, towards the east. Similarly, the geodesic (ψ_2) between $\vec{P}_2$ and $\vec{P}_3$ lies along the equator, from $\vec{P}_2$ towards the east. Patching the two minima curves ψ_1 and ψ_2 together at $\vec{P}_2$ it is observed that, at $\vec{P}_2$, $\psi_1(\vec{P}_2) = \psi_2(\vec{P}_2)$ and $\vec{T}_1(\vec{P}_2) = \vec{T}_2(P_2)$,

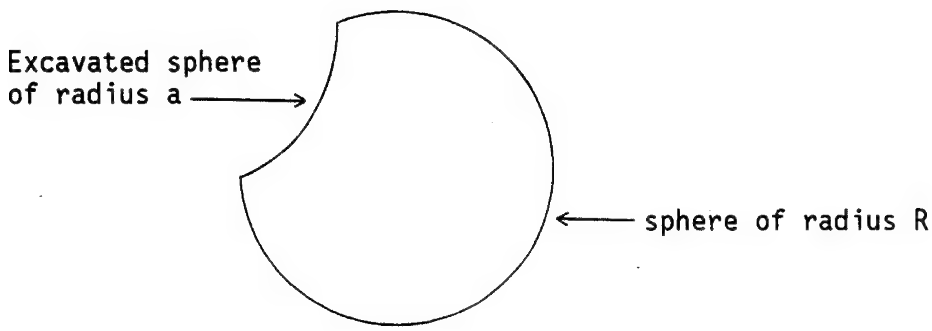


but the true geodesic from $\vec{P}_1$ to $\vec{P}_3$ is <u>not</u> $\psi_1 + \psi_2$.

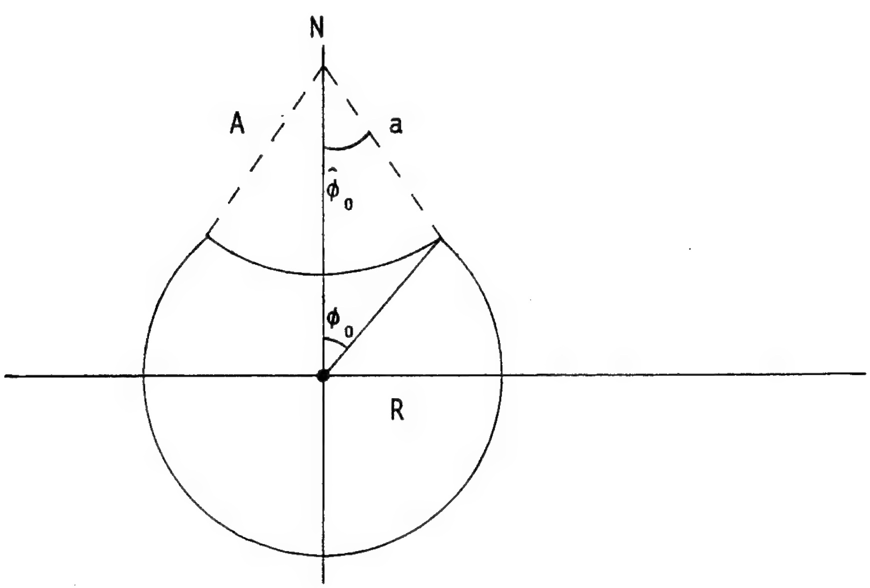
The geodesic from $\vec{P}_1$ to $\vec{P}_3$ starts at $\vec{P}_1$ and goes westward, around the back to $\vec{P}_3$. The solutions for geodesics on more complicated surfaces (such as a composite surfaces) can become very involved and are not necessarily unique. In the next section we will address a simple yet practical problem, solving for the curve of minimum distance on a sphere with a spherical dent feature on its surface. Such a representative surface does exist (approximately) in nature; an example would be the martian moon Phobus.

III A PHOBUS PROTOTYPE COMPOSITE SURFACE NAVIGATION PROBLEM

Consider the following diagram:



Thus, the spherical dent in the outer sphere of radius R is a removed partial sphere of radius a. A useful coordinate system is one which describes the boundary between the two surfaces as a constant.



The outer sphere has its center at the origin, the boundary between the two simple surfaces at $\phi = \phi_0$. The removed sphere having its center at A and corresponding boundary at $\hat{\phi} = \hat{\phi}_0$. Letting the outer sphere be S_1 with geodesic ψ_1 and the removed (inner) sphere be S_2 with geodesic ψ_2, five situations can arise for any two distinct points ($\vec{P}_0$ and $\vec{P}_1$):

a. $\vec{P}_0$ and $\vec{P}_1$ may both reside in S_1,

b. $\vec{P}_0$ and $\vec{P}_1$ may both reside in S_2, or

c. $\vec{P}_0$ and $\vec{P}_1$ in S_1 and S_2 respectively (or vice versa).

d & e. $\vec{P}_0$ or $\vec{P}_1$ or both may reside on the boundary between S_1 & S_2.

Irrespective of the case, the boundary between S_1 and S_2 not being smooth requires us to consider three curves or paths when solving for the curve of minimum distance between points $\vec{P}_0$ and $\vec{P}_1$, these paths are ψ_1, ψ_2* and a third path along the boundary between the two surfaces (at $\phi = \phi_0$, call it ψ_3). Assuming $\vec{P}_0$, $\vec{P}_1$ are unit vectors, then in S_1, the distance between points is $\Delta S_1 = R \cos^{-1} (\vec{P}_0 \cdot \vec{P}_1)$, similarly in S_2, $\Delta S_2 = a \cos^{-1} (\vec{P}_0 \cdot \vec{P}_1)$.

<u>Example 1</u>.

Let $\vec{P}_0$ be on the outer sphere, $\vec{P}_1$ be in the dent. Find the minimum distance curve. Let $\vec{P}(\phi_0, \lambda)$ be on the point on the boundary where the minimum distance curve passes through. The inner and outer spheres are assumed to be longitude aligned, that is, $\vec{P}(\phi_0, \lambda) \equiv \vec{P}(\hat{\phi}_0, \lambda)$. The idea is to solve for λ that minimizes the total distance.

$$\Delta S(\lambda) = R \cos^{-1} \left[\vec{P}_0 \cdot \vec{P}(\phi_0, \lambda) \right] + a \cos^{-1} \left[\vec{P}_1 \cdot \vec{P}(\hat{\phi}_0, \lambda) \right]$$

and set

$$\frac{\partial(\Delta s)}{\partial\lambda} = 0 \quad .$$

*which are great circle curves on the respective spheres, S_1 and S_2.

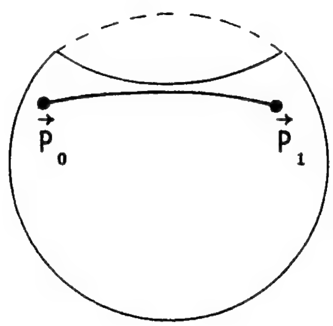
In general, a minima curve between points may cross the boundary twice, at $\vec{P}(\phi, \lambda)$ and $\vec{P}(\phi, u)$. From this, there are several test cases which must be simultaneously checked, solved for and compared to find the true minima.

Example 2.

Let $\vec{P}_0$, $\vec{P}_1$ be on the outer sphere (S_1). Under certain conditions, the minima curve may go into S_2, ride on the border or remain totally in S_1.

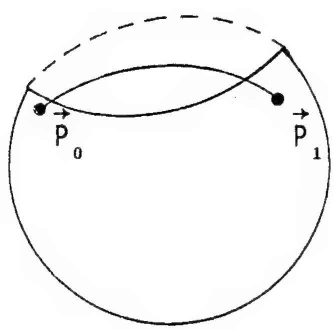
Sample geodesics

I



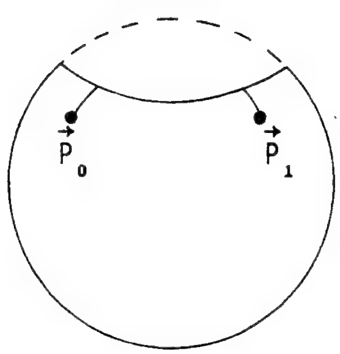
$\psi = \psi_1$

II



$\psi = \psi_1^0 + \psi_2 + \psi_1^1$

III



$\psi = \psi_1^0 + \psi_3 + \psi_1^1$

For each case a test distance is used $\Delta S_1 = R \cos^{-1} (\vec{P}_0 \cdot \vec{P}_1)$ provided all along ψ, $\phi \geq \phi_0$.

$$\Delta S_2 = R \cos^{-1}\left[\vec{P}_0 \cdot P(\phi_0, \lambda)\right] + a \cos^{-1}\left[\vec{P}(\hat{\phi}_0, \lambda) \cdot \vec{P}(\hat{\phi}_0, u)\right] +$$
$$R \cos^{-1}\left[\vec{P}(\phi_0, u) \cdot \vec{P}_1\right]$$

and

$$\frac{\partial \Delta S_2}{\partial \lambda} = \frac{\partial \Delta S_2}{\partial u} = 0 \quad .$$

$$\Delta S_3 = R \cos^{-1}\left[\vec{P}_0 \cdot P(\phi_0, \lambda)\right] + R \sin \phi_0 \cdot [\ \lambda - u\ ] + R \cos^{-1}\left[\vec{P}(\phi_0, u) \cdot \vec{P}_1\right]$$

$$\frac{\partial \Delta S_3}{\partial \lambda} = \frac{\partial \Delta S_3}{\partial u} = 0 \quad .$$

And so forth. Once $P(\phi, \lambda)$, $P(\phi, u)$, etc. are found which satisfy the minimum distance criteria, it is an easy matter to go from $\vec{P}_0$ to $\vec{P}_1$ by way of dividing up the curve and using the appropriate parametric geodesic relation.[5] For navigation on the inside sphere (S_2) it is convenient to think of the S_2 south pole as the S_1 north pole, ie, solve the S_2 portion as though one was navigating at or near the south pole of the excavated sphere, but report the position latitude or $\pi - \hat{\phi}$ or northern hemisphere latitude. In this way the concept of north pole remains consistent, but the curves must be reported separately with respect to each simple surface (inner or outer).

In summary, while not an easy task, it is certainly possible (if not trivial) to examine on a case by case basis, potential geodesics for the minimum distance between points on this simple composite surface, and then solving for the absolute minima. Depending on the positions of A (the excavated or removed sphere center) and the ratio of a to R, it may become possible to make certain generalizations (if $\vec{P}_0$ and $\vec{P}_1$ are in S_2 the geodesic curve may always be ψ_2, and never include ψ_1 or ψ_3 curves; also, if the removed spherical portion is less than a hemisphere, ψ_3, the border or rim geodesic vanishes as a potential solution) thus, each problem must be evaluated on its own merits.

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[5] Carlton-Wippern, op. cit., pp. 38-43.

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AFSPACECOM TECHNICAL NOTE

SURFACE NAVIGATION AND GEODESY

A PARAMETRIC APPROACH

ADDENDUM III

1 JUNE 1989

HEADQUARTERS AIR FORCE SPACE COMMAND

W A R N I N G

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PREFACE

This is the third addendum to AFSPACECOM Technical Note, **Surface Navigation and Geodesy a Parametric Approach**, 1 March 1988. This addendum addresses the derivation of the commonly used navigation curve called the rhumb line or loxodrome.

The loxodrome is a well known curve, constructed so that the angle between any meridian and the loxodrome is constant throughout the curve.[1] Although there are many ways to derive this curve, the derivation presented here is based on analysis provided in the subject technical note[2] and on well known results from general curvilinear analysis. One further property of the loxodrome should be mentioned; when plotted on a Mercator projection the loxodrome becomes a straight line.

[1]The curve of constant compass heading.
[2]Carlton-Wippern, **Surface Navigation and Geodesy a Parametric Approach**, pp 4, 33-35.

CONTENTS

FIGURES

I INTRODUCTION AND GENERAL DERIVATION

Consider u_1, u_2 and u_3 curvilinear coordinates with Jacobian scaling factors h_1, h_2 and h_3.

Then

$$(ds)^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2 \ ,$$

and if u_1, u_2 and u_3 can be written as a function of λ

$\vec{D} = h_1 \frac{du_1}{d\lambda} \hat{e}_1 + h_2 \frac{du_2}{d\lambda} \hat{e}_2 + h_3 \frac{du_3}{d\lambda} \hat{e}_3$,* where $\vec{D}$ is the tangent (or heading) vector.

Assume the surface under consideration can be constructed by setting one of the curvilinear coordinate variables to a constant, u_1= constant (eg, for a sphere $\rho = a$).

The heading angle, measured off one of the remaining principle axes, can be determine by

$$\tan \psi = \frac{h_3 \frac{du_3}{d\lambda}}{h_2 \frac{du_2}{d\lambda}} = \frac{h_3}{h_2} \bullet \frac{du_3}{du_2} \qquad \text{(by chain rule).} \qquad (1)$$

By assuming tan ψ a constant, equation (1) becomes the loxodrome generating differential equation which relates how u_2 is linked with u_3. This equation will be used throughout this addendum. Equation (1) can, in addition to generating a loxodrome curve, be used to generate a general Mercator projection of a surface.

II APPLICATION TO THE PLANAR SURFACE

With

$$s = \int \left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2\right]^{1/2} d\lambda \ ,$$

and

$$s = \int \left[\left(\frac{dz}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda \ ,$$

obtain two loxodrome forms on the plane, both by setting z = constant. For the first case:

*Carlton-Wippern, op.cit. p 4.

$$\tan\psi = \frac{dy}{dx}$$

which solves as

$$y = y_0 + \tan\psi\,(x - x_0) \qquad \text{(a straight line).}$$

For the second case:

$$\tan\psi = r\,\frac{d\theta}{dr}$$

which solves as

$$r = r_0\, e^{\cot\psi(\theta-\theta_0)} \qquad \text{(an exponential spiral).}$$

The differential equation for the second case may appear to the reader to be a familiar relation.[2]

The first case solution (a straight line) is coincidently the planar surface geodesic. A similar analogy exists for the cylinder (where helix is both cylindrical geodesic and loxodrome).

III APPLICATION TO THE CONIC SURFACE

With

$$s = \int \left[\left(\frac{d\rho}{d\lambda}\right)^2 + \rho^2 \sin^2\alpha \left(\frac{d\theta}{d\lambda}\right)^2\right]^{1/2} d\lambda$$

(the surface generated by setting $\phi = \alpha$, a constant), then

$$\tan\psi = \rho \sin\alpha\,\frac{d\theta}{d\rho} \quad ,$$

which solves as

$$\rho = \rho_0\, e^{\cot\psi \bullet \sin\alpha \bullet (\theta-\theta_0)}$$

with ψ oriented as discussed in Surface Navigation and Geodesy.[3] Distance measurement along the conic loxodrome is accomplished by

$$\Delta s = \int \left[1 + \rho^2 \sin^2\alpha \left(\frac{d\theta}{d\rho}\right)^2\right]^{1/2} d\rho$$

$$= \int \left[1 + \tan^2\psi\right]^{1/2} d\rho$$

[2]Thomas, **Calculus and Analytical Geometry**, p. 372, equation 7a.
[3]Carlton-Wippern, op.cit. p 19.

by direct substitution; then:

$$\Delta s = \sec \psi \, \Delta\rho \Big|_{\rho_0}^{\rho} \quad .$$

Singularities occur when $\psi = \pi \ (n \pm \frac{1}{2})$ with $n = 0, \pm 1, \pm 2 \ . \ . \ .$ at which point $\rho = \rho_0$ and

$$\Delta s = \rho_0 \sin \alpha \bullet \Delta\theta \Big|_{\theta_0}^{\theta} \quad .$$

Also of interest is when $\psi = n\pi$ with $n = 0, \pm 1, \pm 2 \ . \ . \ .$ at which point $\theta = \theta_0$

and

$$\Delta s = \Delta\rho \Big|_{\rho_0}^{\rho}$$

IV APPLICATION TO THE SPHERICAL SURFACE

a. Spherical loxodrome.

With

$$\Delta s = \int \left[\rho^2 \left(\frac{d\phi}{d\lambda} \right)^2 + \rho^2 \sin^2 \phi \left(\frac{d\theta}{d\lambda} \right)^2 \right]^{1/2} d\lambda \quad ,$$

and $\rho = a$, the loxodrome is generated by

$$-\tan \psi = \sin \phi \frac{d\theta}{d\phi} \quad ,*$$

from which

$$\int \frac{d\phi}{\sin \phi} = -\cot \psi \bullet (\theta - \theta_0)$$

*The minus sign introduced to reflect the proper orientation, ψ measured from true north; therefore, $d\phi$ becomes $-(d\phi)$ so as to produce a left-handed coordinate system.

$$\ln \tan \frac{\phi}{2} \Bigg|_{\phi_0}^{\phi} = -\cot \psi \bullet (\theta - \theta_0)$$

and

$$\tan \frac{\phi}{2} = \tan \frac{\phi_0}{2} \, e^{-\cot\psi \bullet (\theta - \theta_0)} \quad .$$

For distance measurement along the spherical loxodrome, by direct substitution:

$$\Delta s = a \int [1 + \tan^2 \psi]^{1/2} \, d\phi$$

$$\Delta s = a \sec \psi \, \Delta\phi \Bigg|_{\phi_0}^{\phi} \quad .$$

Again two important singularities arise; $\psi = \pi \left(n \pm \frac{1}{2}\right)$ with $n = 0, \pm 1, \pm 2$ . . . here $\phi = \phi_0$

and

$$\Delta s = a \sin \phi_0 \, \Delta\theta \Bigg|_{\theta_0}^{\theta} \quad ,$$

and $\psi = n\pi$ with $n = 0, \pm 1, \pm 2$ . . ., where $\theta = \theta_0$

and

$$\Delta s = a \bullet \Delta\phi \Bigg|_{\phi_0}^{\phi} \quad .$$

b. Position and velocity vectors measured in inertial space.

By

$$\Delta s = a \sec \psi \, \Delta\phi \Bigg|_{\phi_0}^{\phi}$$

changes in ϕ can be determined, then with

$$\theta = \theta_0 + \tan\psi \bullet \left[\ln\tan\frac{\phi_0}{2} - \ln\tan\frac{\phi}{2}\right] \qquad (\phi \text{ decreasing})$$

changes in θ can be determined. Now

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} \sin\phi\cos[\theta + \Omega(t-\tau)] \\ \sin\phi\sin[\theta + \Omega(t-\tau)] \\ \cos\phi \end{pmatrix}$$

where the term $\Omega(t-\tau)$ has been included to correct for the sphere rotating at an angular velocity Ω, measured from some inertial body centered coordinate system.

For the velocity vector, let

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} \cos\phi\cos\theta\,\dot{\phi} - \sin\phi\sin\theta\,\dot{\theta} \\ \cos\phi\sin\theta\,\dot{\phi} + \sin\phi\cos\theta\,\dot{\theta} \\ -\sin\phi\,\dot{\phi} \end{pmatrix} .$$

ϕ is determined by $\Delta s = -a\sec\psi\,\Delta\phi$ (for ϕ decreasing) $-\dot{s} = a\sec\psi\dot{\phi}$ and letting $\dot{s} = v$ (a constant), then $-\frac{v}{a}\cos\psi = \dot{\phi}$; and with $-\tan\psi = \sin\phi\,\frac{d\theta}{d\phi}$ then using the chain rule $-\tan\psi\,\dot{\phi} = \sin\phi\,\dot{\theta}$, and $\dot{\theta} = \frac{v}{a}\frac{\sin\psi}{\sin\phi}$.

Now, by direct substitution:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = v \begin{pmatrix} -\cos\psi\cos\phi\cos\theta - \sin\psi\sin\theta \\ \sin\psi\cos\theta - \cos\psi\cos\phi\sin\theta \\ \sin\phi\cos\psi \end{pmatrix}$$

We add to this vector the effect of rotation and obtain:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = v \begin{pmatrix} -\cos\psi\cos\phi\cos\bar{\theta} - \sin\psi\sin\bar{\theta} \\ \sin\psi\cos\bar{\theta} - \cos\psi\cos\phi\sin\bar{\theta} \\ \sin\phi\cos\psi \end{pmatrix} + a\Omega \begin{pmatrix} -\sin\phi\sin\bar{\theta} \\ \sin\phi\cos\bar{\theta} \\ 0 \end{pmatrix}$$

where $\bar{\theta} = \theta + \Omega(t - \tau)$.

NOTE: $\vec{R} \bullet \dot{\vec{R}} \equiv 0$ on the static spherical surface, where

$$\vec{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \dot{\vec{R}} = \frac{d\vec{R}}{dt} \quad .$$

V APPLICATION TO THE OBLATE SPHERICAL SURFACE

a. Slightly oblate spheroidal loxodrome.

We start by introducing the oblate spheroidal curvilinear coordinate system:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \xi \begin{pmatrix} \cosh u \sin \nu \cos \theta \\ \cosh u \sin \nu \sin \theta \\ \sinh u \cos \nu \end{pmatrix}$$

This is a curvilinear system which generates oblate spheroids of eccentricity e by setting u to a constant,[4] specifically $u = \operatorname{sech}^{-1} e$. Since u is set constant we will need only find h_2 and h_3 for

$$\Delta s = \xi \int \left[h_2^2 \left(\frac{du_2}{d\lambda} \right)^2 + h_3^2 \left(\frac{du_3}{d\lambda} \right)^2 \right]^{1/2} d\lambda$$

and

$$h_2^2 = \cosh^2 u \cos^2 \nu + \sinh^2 u \sin^2 \nu$$

$$h_3^2 = \cosh^2 u \sin^2 \nu \quad ,$$

so with $\cosh u = \frac{1}{e}$

$$\Delta s = \frac{\xi}{e} \int \left[(1 - e^2 \sin^2 \nu) \left(\frac{d\nu}{d\lambda} \right)^2 + \sin^2 \nu \left(\frac{d\theta}{d\lambda} \right)^2 \right]^{1/2} d\lambda \quad .$$

Now, recognizing $\xi = ae$ (a, the semimajor axis of the ellipsoid) and continuing as before:

$$-\tan \psi = \frac{\sin \nu}{(1 - e^2 \sin^2 \nu)^{1/2}} \bullet \frac{d\theta}{d\nu}$$

[4]Which is equivalent to setting curvilinear (u_1) constant.

$$\frac{(1 - e^2 \sin^2 \nu)^{1/2}}{\sin \nu} d\nu = -\cot \psi \bullet (\theta - \theta_0)$$

$$\tanh^{-1}\left(\frac{\cos \nu}{(1 - e^2 \sin^2 \nu)^{1/2}}\right) - e \tanh^{-1}\left(\frac{e \cos \nu}{(1 - e^2 \sin^2 \nu)^{1/2}}\right)\Bigg|_{\nu_0}^{\nu}$$

$$= -\cot \psi \ (\theta - \theta_0)$$

and with $e^2 << 1$,

$$\frac{1}{\sin \nu} - \frac{e^2}{2} \sin \nu \ d\nu \cong -\cot \psi \bullet (\theta - \theta_0)$$

$$\ln \tan \frac{\nu}{2} + \frac{e^2}{2} \cos \nu \Bigg|_{\nu_0}^{\nu} = - \cot \psi \bullet (\theta - \theta_0) \qquad \text{(for } \nu \text{ decreasing)}$$

Thus, this is the first order correction for ν [which by the way is not ϕ in spherical coordinates, but is related by

$(1 - e^2)^{1/2} \cot \nu = \cot \phi$].

Distance measured along the loxodrome becomes:

$$\Delta s = a \int [1 + \tan^2 \psi]^{1/2} \bullet (1 - e^2 \sin^2 \nu)^{1/2} d\nu$$

$$= a \sec \psi \int (1 - e^2 \sin^2 \nu)^{1/2} d\nu,$$ and with the integrand being

approximately $1 - \frac{e^2}{2} \sin^2 \nu$

$$\Delta s \cong a \sec \psi \left[\Delta\nu - \frac{e^2}{4}\left(\Delta\nu - \frac{1}{2} \sin 2\nu\right)\right] \Bigg|_{\nu_0}^{\nu} .$$

Solution singularities and resolutions are similar to those on the sphere.

b. Position and velocity vectors measured in inertial space.

With Δs determining changes in ν, which then determines the changes in θ, each successive ν, θ is found. The inertial position is determined by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \sin \nu \cos [\theta + \Omega(t - \tau)] \\ a \sin \nu \sin [\theta + \Omega(t - \tau)] \\ b \cos \nu \end{pmatrix}$$

where a, b are the semimajor and semiminor axis, respectively.

For the velocity vector, let

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \cos \nu \cos \theta\, \dot{\nu} - a \sin \nu \sin \theta\, \dot{\theta} \\ a \cos \nu \sin \theta\, \dot{\nu} + a \sin \nu \cos \theta\, \dot{\theta} \\ - b \sin \nu\, \dot{\nu} \end{pmatrix} .$$

As in the spherical case $\dot{\nu}$ is determined by

$$ds = -a \sec \psi\, (1 - e^2 \sin^2 \nu)^{1/2}\, d\nu$$

$$\dot{s} = -a \sec \psi\, (1 - e^2 \sin^2 \nu)^{1/2}\, \dot{\nu} ,$$

and letting as before $\dot{s} = v$, then:

$$\dot{\nu} = -\frac{v}{a} \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2}$$

$$\dot{\theta} = \frac{v}{a} \bullet \frac{\sin \psi}{\sin \nu} ,$$

and by direct substitution

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = v \begin{pmatrix} -\cos \nu \cos \theta \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2} - \sin \theta \sin \psi \\ -\cos \nu \sin \theta \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2} + \cos \theta \sin \psi \\ (1 - e^2)^{1/2} \bullet \sin \nu \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2} \end{pmatrix}$$

and finally, taking into account rotation about the polar (or minor) axis:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = v \begin{pmatrix} -\cos \nu \cos \bar{\theta} \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2} - \sin \bar{\theta} \sin \psi \\ -\cos \nu \sin \bar{\theta} \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2} + \cos \bar{\theta} \sin \psi \\ (1 - e^2)^{1/2} \sin \nu \cos \psi\, (1 - e^2 \sin^2 \nu)^{-1/2} \end{pmatrix}$$

$$+ a\Omega \begin{pmatrix} -\sin \nu \sin \bar{\theta} \\ \sin \nu \cos \bar{\theta} \\ 0 \end{pmatrix}$$

where $\bar{\theta} = \theta + \Omega(t - \tau)$.

VI SURFACES WHERE THE LOXODROME AND GEODESIC CURVES ARE COINCIDENT

Starting with

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2$$

and then setting u_1 = constant, thereby generating a surface spanned by (u_2, u_3);

$$\Delta s = \int \left[1 + \left(\frac{h_3}{h_2}\right)^2 \left(\frac{du_3}{du_2}\right)^2\right]^{1/2} h_2 \, du_2 \quad .$$

Now, letting

$$u_3 = g,$$

$$\frac{du_3}{du_2} = g'$$

and assuming

$$\frac{\partial h_2}{\partial g} = 0 \quad ,$$

$$\frac{\partial h_3}{\partial g} = 0 \quad ,$$

$$\frac{\partial h_2}{\partial g'} = 0 \quad ,$$

and $\dfrac{\partial h_3}{\partial g'} = 0$ $\left(\text{written } \dfrac{\partial(h_2 : h_3)}{\partial(g : g')} = 0\right)$,

the equation which generates the geodesic becomes:

$$\frac{h_2\left(\dfrac{h_3}{h_2}\right)^2 \dfrac{du_3}{du_2}}{\left[1 + \left(\dfrac{h_3}{h_2} \cdot \dfrac{du_3}{du_2}\right)^2\right]^{1/2}} = c$$

But by definition, the loxodrome is generated by

$$\tan \psi = \frac{h_3}{h_2} \frac{du_3}{du_2} = \text{constant.}$$

Solving both equations simultaneously by direct substitution yields:

$$h_3 \frac{\tan \psi}{\sec \psi} = c \quad ,$$

$$h_3 = c \bullet \csc \psi \quad ,$$

and with c, ψ constant, this requires that h_3 = constant.

The geodesic loxodrome curve relation between u_2 and u_3 then becomes:

$$u_3 = u_3^0 + c^{-1} \sin \psi \tan \psi \int h_2 \, du_2 \quad ,$$

where u_3^0 and c^{-1} are the two constants of integration from the (geodesic) second order differential equation.

There are surfaces where the geodesic loxodrome exists (eg, on the plane, the geodesic loxodrome is a straight line; on the cylinder, the geodesic loxodrome is a helix). This mathematical phenomena can occur when by setting u_1 = constant, to generate the surface, in turn implies that:

$$h_3 = \text{constant}$$

and

$$\frac{\partial(h_2 : h_3)}{\partial(g : g')} = 0 \quad .$$

This is not to say that the proposed process of generating the geodesic loxodrome is necessarily unique [$\partial(h_2 : h_3)/\partial(g : g') = 0$ may, in fact, be over restrictive], but that further analysis into the phenomena of the geodesic loxodrome is beyond the scope of this addendum.

On a variety of surfaces (eg, a sphere) a parametric geodesic for a specific case may also be a loxodrome. For example, consider the geodesic between two points on a sphere; one point on the equator, the other at the north pole. The great circle connecting the two points is one of constant heading or direction (due north). In this case the constant c = 0, forcing $\psi = 0$ (or 180°), and the geodesic loxodrome equation reduces to:

$$\theta = \theta_0 \quad ,$$

where θ is the longitudinal variable.

VII SUMMARY AND THE TORUS REVISITED

The loxodrome, while not as efficient a curve as a geodesic for navigation purposes, is a useful navigation tool and is relatively easy to construct on a wide variety of surfaces (provided there exists a set of curvilinear coordinates such that a surface under investigation can be generated by setting one of the coordinates equal to some constant value).

As a final comparison between the tractability of solving for a geodesic or a loxodrome curve, we revisit the torus. Choosing curvilinear coordinates such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} (1 + \nu \sin \phi) \cos \theta \\ (1 + \nu \sin \phi) \sin \theta \\ \nu \cos \phi \end{pmatrix} ,$$

the torus is generated by setting ν = constant, specifically $\nu = \frac{b}{a}$ (see figure 1).

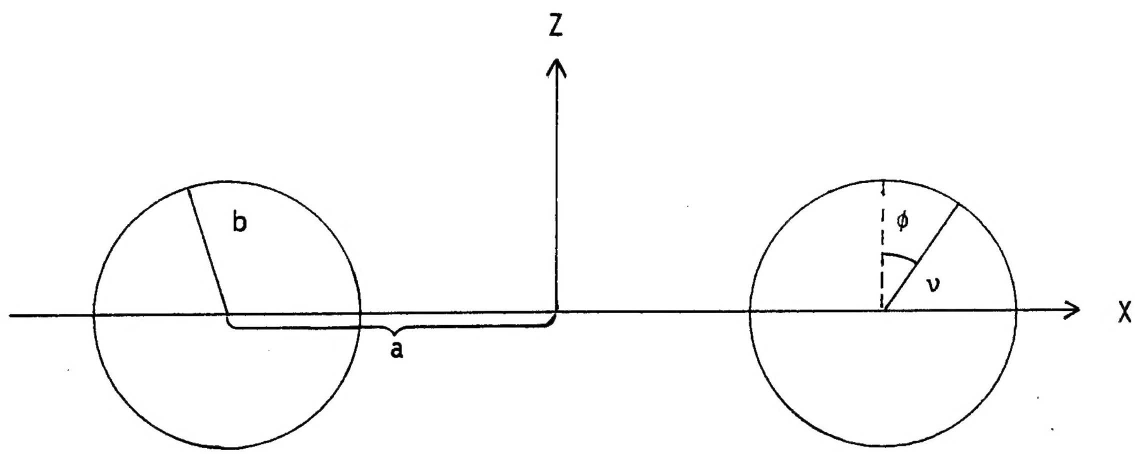

Figure 1. Cross section of a torus

In this coordinate system $h_1 = a$, $h_2 = a\nu$, $h_3 = a(1 + \nu \sin \phi)$, and setting $\nu = \epsilon$, $\epsilon = \frac{b}{a}$ (< 1):

$$\Delta s = a \int \left[\epsilon^2 + (1 + \epsilon \sin \phi)^2 \left(\frac{d\theta}{d\phi} \right)^2 \right]^{1/2} d\phi,$$

then solving for $\theta = \theta(\phi)$,

$$\theta = \theta_0 \pm c\epsilon \int \frac{d\phi}{(1 + \epsilon \sin \phi)[(1 + \epsilon \sin \phi)^2 - c^2]^{1/2}}$$

However, solving for the loxodrome:

$$-\tan \psi = \frac{1 + \epsilon \sin \phi}{\epsilon} \; \frac{d\theta}{d\phi}$$

$$\frac{\epsilon}{1 + \epsilon \sin \phi} \, d\phi = -\cot \psi \; d\theta$$

$$\frac{2\epsilon}{(1 - \epsilon^2)^{1/2}} \cdot \tan^{-1} \left[\frac{\tan\left(\frac{\phi}{2}\right) + \epsilon}{(1 - \epsilon^2)^{1/2}} \right] \Bigg|_{\phi_0}^{\phi} = -\cot \psi \; (\theta - \theta_0)$$

NOTE: The torroidal loxodrome is not only solvable for θ (ie,

$$\theta = \theta_0 - \tan \psi \bullet \frac{2\epsilon}{(1 - \epsilon^2)^{1/2}} \bullet \tan^{-1} \left[\frac{\tan\left(\frac{\phi}{2}\right) + \epsilon}{(1 - \epsilon^2)^{1/2}} \right] \Bigg|_{\phi_0}^{\phi} \Bigg)$$

but also invertable so as to make $\phi = \phi(\theta)$; given ϕ_0, θ_0 (ie, calling

$$\hat{\phi}_0 = \frac{2\epsilon}{(1 - \epsilon^2)^{1/2}} \bullet \tan^{-1} \left[\frac{\tan\left(\frac{\phi_0}{2}\right) + \epsilon}{(1 - \epsilon^2)^{1/2}} \right]$$

$$\phi = 2 \tan^{-1} \left\{ (1 - \epsilon^2)^{1/2} \bullet \tan \left[\frac{(1 - e^2)^{1/2}}{2\epsilon} \left(\hat{\phi}_0 - \cot \psi \; (\theta - \theta_0)\right) \right] - \epsilon \right\} \Bigg) \quad ,$$

and the measurement of distance along the torroidal loxodrome becomes:

$$\Delta s = a \int \left[\epsilon^2 + (1 + \epsilon \sin \phi)^2 \left(\frac{d\theta}{d\phi} \right)^2 \right]^{1/2} d\phi$$

$$\Delta s = b \int \left[1 + \left(\frac{1 + \epsilon \sin \phi}{\epsilon} \bullet \frac{d\theta}{d\phi} \right)^2 \right]^{1/2} d\phi \qquad \left(\text{remembering } \epsilon = \frac{b}{a} \right)$$

$$\Delta s = b \sec \psi \; \Delta\phi \quad .$$

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TECHNICAL NOTE

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